

## NONHOMOGENEOUS LINEAR EQUATIONS

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$1 \quad ay'' + by' + cy = G(x)$$

where  $a$ ,  $b$ , and  $c$  are constants and  $G$  is a continuous function. The related homogeneous equation

$$2 \quad ay'' + by' + cy = 0$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

**3 Theorem** The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where  $y_p$  is a particular solution of Equation 1 and  $y_c$  is the general solution of the complementary Equation 2.

**Proof** All we have to do is verify that if  $y$  is any solution of Equation 1, then  $y - y_p$  is a solution of the complementary Equation 2. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= g(x) - g(x) = 0 \end{aligned}$$

We know from *Additional Topics: Second-Order Linear Differential Equations* how to solve the complementary equation. (Recall that the solution is  $y_c = c_1y_1 + c_2y_2$ , where  $y_1$  and  $y_2$  are linearly independent solutions of Equation 2.) Therefore, Theorem 3 says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution  $y_p$ . There are two methods for finding a particular solution: The method of undetermined coefficients is straightforward but works only for a restricted class of functions  $G$ . The method of variation of parameters works for every function  $G$  but is usually more difficult to apply in practice.

### THE METHOD OF UNDETERMINED COEFFICIENTS

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where  $G(x)$  is a polynomial. It is reasonable to guess that there is a particular solution  $y_p$  that is a polynomial of the same degree as  $G$  because if  $y$  is a polynomial, then  $ay'' + by' + cy$  is also a polynomial. We therefore substitute  $y_p(x) =$  a polynomial (of the same degree as  $G$ ) into the differential equation and determine the coefficients.

**EXAMPLE 1** Solve the equation  $y'' + y' - 2y = x^2$ .

**SOLUTION** The auxiliary equation of  $y'' + y' - 2y = 0$  is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots  $r = 1, -2$ . So the solution of the complementary equation is

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Since  $G(x) = x^2$  is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then  $y_p' = 2Ax + B$  and  $y_p'' = 2A$  so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or 
$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 1 \quad 2A - 2B = 0 \quad 2A + B - 2C = 0$$

The solution of this system of equations is

$$A = -\frac{1}{2} \quad B = -\frac{1}{2} \quad C = -\frac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

■ Figure 1 shows four solutions of the differential equation in Example 1 in terms of the particular solution  $y_p$  and the functions  $f(x) = e^x$  and  $g(x) = e^{-2x}$ .

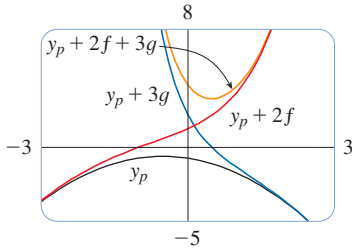


FIGURE 1

If  $G(x)$  (the right side of Equation 1) is of the form  $Ce^{kx}$ , where  $C$  and  $k$  are constants, then we take as a trial solution a function of the same form,  $y_p(x) = Ae^{kx}$ , because the derivatives of  $e^{kx}$  are constant multiples of  $e^{kx}$ .

**EXAMPLE 2** Solve  $y'' + 4y = e^{3x}$ .

**SOLUTION** The auxiliary equation is  $r^2 + 4 = 0$  with roots  $\pm 2i$ , so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try  $y_p(x) = Ae^{3x}$ . Then  $y_p' = 3Ae^{3x}$  and  $y_p'' = 9Ae^{3x}$ . Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so  $13Ae^{3x} = e^{3x}$  and  $A = \frac{1}{13}$ . Thus, a particular solution is

$$y_p(x) = \frac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{13}e^{3x}$$

■ Figure 2 shows solutions of the differential equation in Example 2 in terms of  $y_p$  and the functions  $f(x) = \cos 2x$  and  $g(x) = \sin 2x$ . Notice that all solutions approach  $\infty$  as  $x \rightarrow \infty$  and all solutions resemble sine functions when  $x$  is negative.

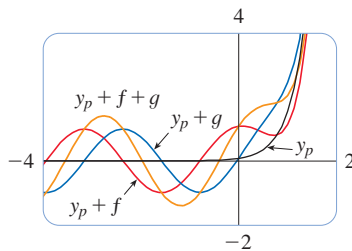


FIGURE 2

If  $G(x)$  is either  $C \cos kx$  or  $C \sin kx$ , then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A \cos kx + B \sin kx$$

**EXAMPLE 3** Solve  $y'' + y' - 2y = \sin x$ .

**SOLUTION** We try a particular solution

$$y_p(x) = A \cos x + B \sin x$$

Then  $y'_p = -A \sin x + B \cos x$        $y''_p = -A \cos x - B \sin x$

so substitution in the differential equation gives

$$(-A \cos x - B \sin x) + (-A \sin x + B \cos x) - 2(A \cos x + B \sin x) = \sin x$$

or  $(-3A + B) \cos x + (-A - 3B) \sin x = \sin x$

This is true if

$$-3A + B = 0 \quad \text{and} \quad -A - 3B = 1$$

The solution of this system is

$$A = -\frac{1}{10} \quad B = -\frac{3}{10}$$

so a particular solution is

$$y_p(x) = -\frac{1}{10} \cos x - \frac{3}{10} \sin x$$

In Example 1 we determined that the solution of the complementary equation is  $y_c = c_1 e^x + c_2 e^{-2x}$ . Thus, the general solution of the given equation is

$$y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{10}(\cos x + 3 \sin x) \quad \blacksquare \blacksquare$$

If  $G(x)$  is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x \cos 3x$$

we would try

$$y_p(x) = (Ax + B) \cos 3x + (Cx + D) \sin 3x$$

If  $G(x)$  is a sum of functions of these types, we use the easily verified *principle of superposition*, which says that if  $y_{p_1}$  and  $y_{p_2}$  are solutions of

$$ay'' + by' + cy = G_1(x) \quad ay'' + by' + cy = G_2(x)$$

respectively, then  $y_{p_1} + y_{p_2}$  is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

**EXAMPLE 4** Solve  $y'' - 4y = xe^x + \cos 2x$ .

**SOLUTION** The auxiliary equation is  $r^2 - 4 = 0$  with roots  $\pm 2$ , so the solution of the complementary equation is  $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$ . For the equation  $y'' - 4y = xe^x$  we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then  $y'_{p_1} = (Ax + A + B)e^x$ ,  $y''_{p_1} = (Ax + 2A + B)e^x$ , so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or  $(-3Ax + 2A - 3B)e^x = xe^x$

Thus,  $-3A = 1$  and  $2A - 3B = 0$ , so  $A = -\frac{1}{3}$ ,  $B = -\frac{2}{9}$ , and

$$y_{p_1}(x) = \left(-\frac{1}{3}x - \frac{2}{9}\right)e^x$$

For the equation  $y'' - 4y = \cos 2x$ , we try

$$y_{p_2}(x) = C \cos 2x + D \sin 2x$$

Substitution gives

$$-4C \cos 2x - 4D \sin 2x - 4(C \cos 2x + D \sin 2x) = \cos 2x$$

or

$$-8C \cos 2x - 8D \sin 2x = \cos 2x$$

Therefore,  $-8C = 1$ ,  $-8D = 0$ , and

$$y_{p_2}(x) = -\frac{1}{8} \cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1 e^{2x} + c_2 e^{-2x} - \left(\frac{1}{3}x + \frac{2}{9}\right)e^x - \frac{1}{8} \cos 2x$$

■ ■ In Figure 3 we show the particular solution  $y_p = y_{p_1} + y_{p_2}$  of the differential equation in Example 4. The other solutions are given in terms of  $f(x) = e^{2x}$  and  $g(x) = e^{-2x}$ .

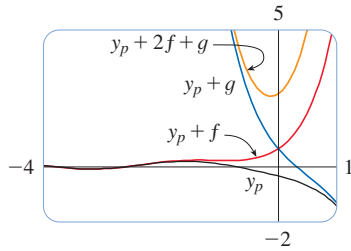


FIGURE 3

Finally we note that the recommended trial solution  $y_p$  sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by  $x$  (or by  $x^2$  if necessary) so that no term in  $y_p(x)$  is a solution of the complementary equation.

**EXAMPLE 5** Solve  $y'' + y = \sin x$ .

**SOLUTION** The auxiliary equation is  $r^2 + 1 = 0$  with roots  $\pm i$ , so the solution of the complementary equation is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

Ordinarily, we would use the trial solution

$$y_p(x) = A \cos x + B \sin x$$

but we observe that it is a solution of the complementary equation, so instead we try

$$y_p(x) = Ax \cos x + Bx \sin x$$

Then

$$y_p'(x) = A \cos x - Ax \sin x + B \sin x + Bx \cos x$$

$$y_p''(x) = -2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x$$

Substitution in the differential equation gives

$$y_p'' + y_p = -2A \sin x + 2B \cos x = \sin x$$

so  $A = -\frac{1}{2}$ ,  $B = 0$ , and

$$y_p(x) = -\frac{1}{2}x \cos x$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \cos x$$

■ ■ The graphs of four solutions of the differential equation in Example 5 are shown in Figure 4.

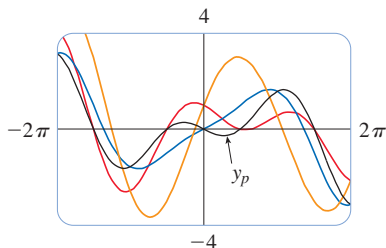


FIGURE 4

We summarize the method of undetermined coefficients as follows:

1. If  $G(x) = e^{kx}P(x)$ , where  $P$  is a polynomial of degree  $n$ , then try  $y_p(x) = e^{kx}Q(x)$ , where  $Q(x)$  is an  $n$ th-degree polynomial (whose coefficients are determined by substituting in the differential equation.)
2. If  $G(x) = e^{kx}P(x) \cos mx$  or  $G(x) = e^{kx}P(x) \sin mx$ , where  $P$  is an  $n$ th-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x) \cos mx + e^{kx}R(x) \sin mx$$

where  $Q$  and  $R$  are  $n$ th-degree polynomials.

*Modification:* If any term of  $y_p$  is a solution of the complementary equation, multiply  $y_p$  by  $x$  (or by  $x^2$  if necessary).

**EXAMPLE 6** Determine the form of the trial solution for the differential equation  $y'' - 4y' + 13y = e^{2x} \cos 3x$ .

**SOLUTION** Here  $G(x)$  has the form of part 2 of the summary, where  $k = 2$ ,  $m = 3$ , and  $P(x) = 1$ . So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A \cos 3x + B \sin 3x)$$

But the auxiliary equation is  $r^2 - 4r + 13 = 0$ , with roots  $r = 2 \pm 3i$ , so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

This means that we have to multiply the suggested trial solution by  $x$ . So, instead, we use

$$y_p(x) = xe^{2x}(A \cos 3x + B \sin 3x) \quad \blacksquare$$

## THE METHOD OF VARIATION OF PARAMETERS

Suppose we have already solved the homogeneous equation  $ay'' + by' + cy = 0$  and written the solution as

$$\boxed{4} \quad y(x) = c_1y_1(x) + c_2y_2(x)$$

where  $y_1$  and  $y_2$  are linearly independent solutions. Let's replace the constants (or parameters)  $c_1$  and  $c_2$  in Equation 4 by arbitrary functions  $u_1(x)$  and  $u_2(x)$ . We look for a particular solution of the nonhomogeneous equation  $ay'' + by' + cy = G(x)$  of the form

$$\boxed{5} \quad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

(This method is called **variation of parameters** because we have varied the parameters  $c_1$  and  $c_2$  to make them functions.) Differentiating Equation 5, we get

$$\boxed{6} \quad y'_p = (u'_1y_1 + u'_2y_2) + (u_1y'_1 + u_2y'_2)$$

Since  $u_1$  and  $u_2$  are arbitrary functions, we can impose two conditions on them. One condition is that  $y_p$  is a solution of the differential equation; we can choose the other condition so as to simplify our calculations. In view of the expression in Equation 6, let's impose the condition that

$$\boxed{7} \quad u'_1y_1 + u'_2y_2 = 0$$

$$\text{Then} \quad y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Substituting in the differential equation, we get

$$a(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + b(u_1 y_1' + u_2 y_2') + c(u_1 y_1 + u_2 y_2) = G$$

or

$$\mathbf{8} \quad u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1' y_1' + u_2' y_2') = G$$

But  $y_1$  and  $y_2$  are solutions of the complementary equation, so

$$ay_1'' + by_1' + cy_1 = 0 \quad \text{and} \quad ay_2'' + by_2' + cy_2 = 0$$

and Equation 8 simplifies to

$$\mathbf{9} \quad a(u_1' y_1' + u_2' y_2') = G$$

Equations 7 and 9 form a system of two equations in the unknown functions  $u_1'$  and  $u_2'$ . After solving this system we may be able to integrate to find  $u_1$  and  $u_2$  and then the particular solution is given by Equation 5.

**EXAMPLE 7** Solve the equation  $y'' + y = \tan x$ ,  $0 < x < \pi/2$ .

**SOLUTION** The auxiliary equation is  $r^2 + 1 = 0$  with roots  $\pm i$ , so the solution of  $y'' + y = 0$  is  $c_1 \sin x + c_2 \cos x$ . Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x) \sin x + u_2(x) \cos x$$

$$\text{Then} \quad y_p' = (u_1' \sin x + u_2' \cos x) + (u_1 \cos x - u_2 \sin x)$$

Set

$$\mathbf{10} \quad u_1' \sin x + u_2' \cos x = 0$$

$$\text{Then} \quad y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For  $y_p$  to be a solution we must have

$$\mathbf{11} \quad y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

$$u_1' = \sin x \quad u_1(x) = -\cos x$$

(We seek a particular solution, so we don't need a constant of integration here.) Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x} \quad u_2' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$\text{So} \quad u_2(x) = \sin x - \ln(\sec x + \tan x)$$

■ Figure 5 shows four solutions of the differential equation in Example 7.

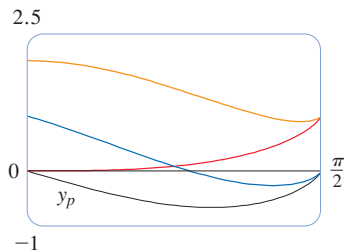


FIGURE 5

(Note that  $\sec x + \tan x > 0$  for  $0 < x < \pi/2$ .) Therefore

$$\begin{aligned} y_p(x) &= -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x \\ &= -\cos x \ln(\sec x + \tan x) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$


**EXERCISES**

**A** [Click here for answers.](#)

**S** [Click here for solutions.](#)

**1–10** ■ Solve the differential equation or initial-value problem using the method of undetermined coefficients.

- |  |                              |
|--|------------------------------|
| 1. $y'' + 3y' + 2y = x^2$  | 2. $y'' + 9y = e^{3x}$       |
| 3. $y'' - 2y' = \sin 4x$   | 4. $y'' + 6y' + 9y = 1 + x$  |
| 5. $y'' - 4y' + 5y = e^{-x}$                                       | 6. $y'' + 2y' + y = xe^{-x}$ |
| 7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$          |                              |
| 8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$        |                              |
| 9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$              |                              |
| 10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$ |                              |

 **11–12** ■ Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11.  $4y'' + 5y' + y = e^x$   
 12.  $2y'' + 3y' + y = 1 + \cos 2x$

**13–18** ■ Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13.  $y'' + 9y = e^{2x} + x^2 \sin x$   
 14.  $y'' + 9y' = xe^{-x} \cos \pi x$   
 15.  $y'' + 9y' = 1 + xe^{9x}$

16.  $y'' + 3y' - 4y = (x^3 + x)e^x$   
 17.  $y'' + 2y' + 10y = x^2e^{-x} \cos 3x$   
 18.  $y'' + 4y = e^{3x} + x \sin 2x$

**19–22** ■ Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

19.  $y'' + 4y = x$   
 20.  $y'' - 3y' + 2y = \sin x$   
 21.  $y'' - 2y' + y = e^{2x}$   
 22.  $y'' - y' = e^x$

**23–28** ■ Solve the differential equation using the method of variation of parameters.

23.  $y'' + y = \sec x, \quad 0 < x < \pi/2$   
 24.  $y'' + y = \cot x, \quad 0 < x < \pi/2$   
 25.  $y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}$   
 26.  $y'' + 3y' + 2y = \sin(e^x)$   
 27.  $y'' - y = \frac{1}{x}$   
 28.  $y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$

## ANSWERS

[S Click here for solutions.](#)

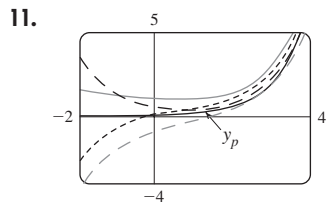
1.  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$

3.  $y = c_1 + c_2 e^{2x} + \frac{1}{40} \cos 4x - \frac{1}{20} \sin 4x$

5.  $y = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$

7.  $y = \frac{3}{2} \cos x + \frac{11}{2} \sin x + \frac{1}{2}e^x + x^3 - 6x$

9.  $y = e^x(\frac{1}{2}x^2 - x + 2)$



The solutions are all asymptotic to  $y_p = e^x/10$  as  $x \rightarrow \infty$ . Except for  $y_p$ , all solutions approach either  $\infty$  or  $-\infty$  as  $x \rightarrow -\infty$ .

13.  $y_p = Ae^{2x} + (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x$

15.  $y_p = Ax + (Bx + C)e^{9x}$

17.  $y_p = xe^{-x}[(Ax^2 + Bx + C) \cos 3x + (Dx^2 + Ex + F) \sin 3x]$

19.  $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$

21.  $y = c_1 e^x + c_2 x e^x + e^{2x}$

23.  $y = (c_1 + x) \sin x + (c_2 + \ln \cos x) \cos x$

25.  $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$

27.  $y = [c_1 - \frac{1}{2} \int (e^x/x) dx]e^{-x} + [c_2 + \frac{1}{2} \int (e^{-x}/x) dx]e^x$



**SOLUTIONS**

1. The auxiliary equation is  $r^2 + 3r + 2 = (r + 2)(r + 1) = 0$ , so the complementary solution is

$$y_c(x) = c_1 e^{-2x} + c_2 e^{-x}. \text{ We try the particular solution } y_p(x) = Ax^2 + Bx + C, \text{ so } y'_p = 2Ax + B \text{ and } y''_p = 2A.$$

Substituting into the differential equation, we have  $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$  or

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2. \text{ Comparing coefficients gives } 2A = 1, 6A + 2B = 0, \text{ and}$$

$$2A + 3B + 2C = 0, \text{ so } A = \frac{1}{2}, B = -\frac{3}{2}, \text{ and } C = \frac{7}{4}. \text{ Thus the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

3. The auxiliary equation is  $r^2 - 2r = r(r - 2) = 0$ , so the complementary solution is  $y_c(x) = c_1 + c_2 e^{2x}$ . Try the

$$\text{particular solution } y_p(x) = A \cos 4x + B \sin 4x, \text{ so } y'_p = -4A \sin 4x + 4B \cos 4x$$

$$\text{and } y''_p = -16A \cos 4x - 16B \sin 4x. \text{ Substitution into the differential equation}$$

$$\text{gives } (-16A \cos 4x - 16B \sin 4x) - 2(-4A \sin 4x + 4B \cos 4x) = \sin 4x \Rightarrow$$

$$(-16A - 8B) \cos 4x + (8A - 16B) \sin 4x = \sin 4x. \text{ Then } -16A - 8B = 0 \text{ and } 8A - 16B = 1 \Rightarrow A = \frac{1}{40}$$

$$\text{and } B = -\frac{1}{20}. \text{ Thus the general solution is } y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{2x} + \frac{1}{40} \cos 4x - \frac{1}{20} \sin 4x.$$

5. The auxiliary equation is  $r^2 - 4r + 5 = 0$  with roots  $r = 2 \pm i$ , so the complementary solution is

$$y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x). \text{ Try } y_p(x) = Ae^{-x}, \text{ so } y'_p = -Ae^{-x} \text{ and } y''_p = Ae^{-x}. \text{ Substitution gives}$$

$$Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}. \text{ Thus the general solution is}$$

$$y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}.$$

7. The auxiliary equation is  $r^2 + 1 = 0$  with roots  $r = \pm i$ , so the complementary solution is

$$y_c(x) = c_1 \cos x + c_2 \sin x. \text{ For } y'' + y = e^x \text{ try } y_{p1}(x) = Ae^x. \text{ Then } y'_{p1} = y''_{p1} = Ae^x \text{ and substitution gives}$$

$$Ae^x + Ae^x = e^x \Rightarrow A = \frac{1}{2}, \text{ so } y_{p1}(x) = \frac{1}{2}e^x. \text{ For } y'' + y = x^3 \text{ try } y_{p2}(x) = Ax^3 + Bx^2 + Cx + D.$$

$$\text{Then } y'_{p2} = 3Ax^2 + 2Bx + C \text{ and } y''_{p2} = 6Ax + 2B. \text{ Substituting, we have}$$

$$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3, \text{ so } A = 1, B = 0, 6A + C = 0 \Rightarrow C = -6, \text{ and } 2B + D = 0 \Rightarrow$$

$$D = 0. \text{ Thus } y_{p2}(x) = x^3 - 6x \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + x^3 - 6x. \text{ But } 2 = y(0) = c_1 + \frac{1}{2} \Rightarrow c_1 = \frac{3}{2}$$

$$\text{and } 0 = y'(0) = c_2 + \frac{1}{2} - 6 \Rightarrow c_2 = \frac{11}{2}. \text{ Thus the solution to the initial-value problem is}$$

$$y(x) = \frac{3}{2} \cos x + \frac{11}{2} \sin x + \frac{1}{2}e^x + x^3 - 6x.$$

9. The auxiliary equation is  $r^2 - r = 0$  with roots  $r = 0, r = 1$  so the complementary solution is  $y_c(x) = c_1 + c_2 e^x$ .

Try  $y_p(x) = x(Ax + B)e^x$  so that no term in  $y_p$  is a solution of the complementary equation. Then

$$y'_p = (Ax^2 + (2A + B)x + B)e^x \text{ and } y''_p = (Ax^2 + (4A + B)x + (2A + 2B))e^x. \text{ Substitution into the differential}$$

$$\text{equation gives } (Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \Rightarrow$$

$$(2Ax + (2A + B))e^x = xe^x \Rightarrow A = \frac{1}{2}, B = -1. \text{ Thus } y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x \text{ and the general solution is}$$

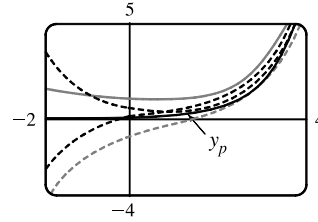
$$y(x) = c_1 + c_2 e^x + \left(\frac{1}{2}x^2 - x\right)e^x. \text{ But } 2 = y(0) = c_1 + c_2 \text{ and } 1 = y'(0) = c_2 - 1, \text{ so } c_2 = 2 \text{ and } c_1 = 0. \text{ The}$$

$$\text{solution to the initial-value problem is } y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right).$$

11.  $y_c(x) = c_1 e^{-x/4} + c_2 e^{-x}$ . Try  $y_p(x) = Ae^x$ . Then

$10Ae^x = e^x$ , so  $A = \frac{1}{10}$  and the general solution is

$y(x) = c_1 e^{-x/4} + c_2 e^{-x} + \frac{1}{10} e^x$ . The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as  $x \rightarrow -\infty$ ), they all approach either  $\infty$  or  $-\infty$  as  $x \rightarrow -\infty$ . As  $x \rightarrow \infty$ , all solutions are asymptotic to  $y_p = \frac{1}{10} e^x$ .



13. Here  $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$ . For  $y'' + 9y = e^{2x}$  try  $y_{p1}(x) = Ae^{2x}$  and for  $y'' + 9y = x^2 \sin x$  try

$y_{p2}(x) = (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x$ . Thus a trial solution is

$y_p(x) = y_{p1}(x) + y_{p2}(x) = Ae^{2x} + (Bx^2 + Cx + D) \cos x + (Ex^2 + Fx + G) \sin x$ .

15. Here  $y_c(x) = c_1 + c_2 e^{-9x}$ . For  $y'' + 9y' = 1$  try  $y_{p1}(x) = Ax$  (since  $y = A$  is a solution to the complementary equation) and for  $y'' + 9y' = xe^{9x}$  try  $y_{p2}(x) = (Bx + C)e^{9x}$ .

17. Since  $y_c(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$  we try

$y_p(x) = x(Ax^2 + Bx + C)e^{-x} \cos 3x + x(Dx^2 + Ex + F)e^{-x} \sin 3x$  (so that no term of  $y_p$  is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u'_1 = -\frac{Gy_2}{a(y_1y'_2 - y_2y'_1)} \quad \text{and} \quad u'_2 = \frac{Gy_1}{a(y_1y'_2 - y_2y'_1)}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) The complementary solution is  $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$ . A particular solution is of the form

$y_p(x) = Ax + B$ . Thus,  $4Ax + 4B = x \Rightarrow A = \frac{1}{4}$  and  $B = 0 \Rightarrow y_p(x) = \frac{1}{4}x$ . Thus, the general solution is  $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$ .

(b) In (a),  $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$ , so set  $y_1 = \cos 2x$ ,  $y_2 = \sin 2x$ . Then

$$y_1y'_2 - y_2y'_1 = 2 \cos^2 2x + 2 \sin^2 2x = 2 \text{ so } u'_1 = -\frac{1}{2}x \sin 2x \Rightarrow$$

$$u_1(x) = -\frac{1}{2} \int x \sin 2x \, dx = -\frac{1}{4}(-x \cos 2x + \frac{1}{2} \sin 2x) \text{ [by parts] and } u'_2 = \frac{1}{2}x \cos 2x$$

$$\Rightarrow u_2(x) = \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{4}(x \sin 2x + \frac{1}{2} \cos 2x) \text{ [by parts]. Hence}$$

$$y_p(x) = -\frac{1}{4}(-x \cos 2x + \frac{1}{2} \sin 2x) \cos 2x + \frac{1}{4}(x \sin 2x + \frac{1}{2} \cos 2x) \sin 2x = \frac{1}{4}x. \text{ Thus}$$

$$y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x.$$

21. (a)  $r^2 - r = r(r - 1) = 0 \Rightarrow r = 0, 1$ , so the complementary solution is  $y_c(x) = c_1 e^x + c_2 x e^x$ . A particular solution is of the form  $y_p(x) = Ae^{2x}$ . Thus  $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \Rightarrow Ae^{2x} = e^{2x} \Rightarrow A = 1$

$\Rightarrow y_p(x) = e^{2x}$ . So a general solution is  $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}$ .

(b) From (a),  $y_c(x) = c_1 e^x + c_2 x e^x$ , so set  $y_1 = e^x$ ,  $y_2 = x e^x$ . Then,  $y_1y'_2 - y_2y'_1 = e^{2x}(1 + x) - x e^{2x} = e^{2x}$

$$\text{and so } u'_1 = -x e^x \Rightarrow u_1(x) = -\int x e^x \, dx = -(x - 1)e^x \text{ [by parts] and } u'_2 = e^x \Rightarrow$$

$$u_2(x) = \int e^x \, dx = e^x. \text{ Hence } y_p(x) = (1 - x)e^{2x} + x e^{2x} = e^{2x} \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}.$$

23. As in Example 6,  $y_c(x) = c_1 \sin x + c_2 \cos x$ , so set  $y_1 = \sin x$ ,  $y_2 = \cos x$ . Then

$$y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1, \text{ so } u_1' = -\frac{\sec x \cos x}{-1} = 1 \Rightarrow u_1(x) = x \text{ and}$$

$$u_2' = \frac{\sec x \sin x}{-1} = -\tan x \Rightarrow u_2(x) = -\int \tan x dx = \ln |\cos x| = \ln(\cos x) \text{ on } 0 < x < \frac{\pi}{2}. \text{ Hence}$$

$$y_p(x) = x \sin x + \cos x \ln(\cos x) \text{ and the general solution is } y(x) = (c_1 + x) \sin x + [c_2 + \ln(\cos x)] \cos x.$$

25.  $y_1 = e^x$ ,  $y_2 = e^{2x}$  and  $y_1 y_2' - y_2 y_1' = e^{3x}$ . So  $u_1' = \frac{-e^{2x}}{(1 + e^{-x})e^{3x}} = -\frac{e^{-x}}{1 + e^{-x}}$  and

$$u_1(x) = \int -\frac{e^{-x}}{1 + e^{-x}} dx = \ln(1 + e^{-x}). \quad u_2' = \frac{e^x}{(1 + e^{-x})e^{3x}} = \frac{e^x}{e^{3x} + e^{2x}} \text{ so}$$

$$u_2(x) = \int \frac{e^x}{e^{3x} + e^{2x}} dx = \ln\left(\frac{e^x + 1}{e^x}\right) - e^{-x} = \ln(1 + e^{-x}) - e^{-x}. \text{ Hence}$$

$$y_p(x) = e^x \ln(1 + e^{-x}) + e^{2x} [\ln(1 + e^{-x}) - e^{-x}] \text{ and the general solution is}$$

$$y(x) = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}.$$

27.  $y_1 = e^{-x}$ ,  $y_2 = e^x$  and  $y_1 y_2' - y_2 y_1' = 2$ . So  $u_1' = -\frac{e^x}{2x}$ ,  $u_2' = \frac{e^{-x}}{2x}$  and

$$y_p(x) = -e^{-x} \int \frac{e^x}{2x} dx + e^x \int \frac{e^{-x}}{2x} dx. \text{ Hence the general solution is}$$

$$y(x) = \left(c_1 - \int \frac{e^x}{2x} dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x} dx\right)e^x.$$