Harvard College

Math 21a: Multivariable Calculus

Formula and Theorem Review

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9 Vectors and the Geometry of Space

9.1 Distance Formula in 3 Dimensions

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

9.2 Equation of a Sphere

The equation of a sphere with center (h, k, l) and radius r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

9.3 Properties of Vectors

If $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors and c and d are scalars:

$$\begin{array}{ll} \vec{a} + \vec{b} = \vec{b} + \vec{a} & \vec{a} + 0 = \vec{a} \\ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} & \vec{a} + -\vec{a} = 0 \\ c(\vec{a} + \vec{b}) = c\vec{a} + c + \vec{b} & (c + d)\vec{a} = c\vec{a} + d\vec{a} \\ (cd)\vec{a} = c(d\vec{a}) & \end{array}$$

9.4 Unit Vector

A unit vector is a vector whose length is 1. The unit vector $\vec{u}$ in the same direction as $\vec{a}$ is given by:

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

9.5 Dot Product

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

9.6 Properties of the Dot Product

Two vectors are orthogonal if their dot product is 0.

$$\begin{array}{ll} \vec{a} \cdot \vec{a} = |\vec{a}|^2 & \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \\ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} & (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b}) \\ 0 \cdot \vec{a} = 0 & \end{array}$$

9.7 Vector Projections

Scalar projection of $\vec{b}$ onto $\vec{a}$:

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

9.8 Cross Product

$$\vec{a}\times\vec{b} = (|\vec{a}||\vec{b}|\sin\theta)\vec{n}$$

where $\vec{n}$ is the unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

9.9 Properties of the Cross Product

Two vectors are parallel if their cross product is 0.

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$

$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

9.10 Scalar Triple Product

The volume of the parallelpiped determined by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

9.11 Vector Equation of a Line

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

9.12 Symmetric Equations of a Line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

where the vector $\vec{c} = \langle a, b, c\rangle$ is the direction of the line.

The symmetric equations for a line passing through the points (x_0, y_0, z_0) and (x_1, y_1, z_1) are given by:

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

9.13 Segment of a Line

The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad \text{for} \quad 0 \leq t \leq 1$$

9.14 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where $\vec{n}$ is the vector orthogonal to every vector in the given plane and $\vec{r} - \vec{r}_0$ is the vector between any two points on the plane.

9.15 Scalar Equation of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and $\langle a, b, c\rangle$ is the vector normal to the plane.

9.16 Distance Between Point and Plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

where P is a point, Σ is a plane, Q is a point on plane Σ, and $\vec{n}$ is the vector orthogonal to the plane.

9.17 Distance Between Point and Line

$$d(P, L) = \frac{|\vec{PQ} \times \vec{u}|}{|\vec{u}|}$$

where P is a point in space, Q is a point on the line L, and $\vec{u}$ is the direction of line.

9.18 Distance Between Line and Line

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

where P is a point on line L, Q is a point on line M, $\vec{u}$ is the direction of line L, and $\vec{v}$ is the direction of line M.

9.19 Distance Between Plane and Plane

$$d = \frac{|e - d|}{|\vec{n}|}$$

where $\vec{n}$ is the vector orthogonal to both planes, e is the constant of one plane, and d is the constant of the other. The distance between non-parallel planes is 0.

9.20 Quadric Surfaces

$$\begin{aligned}
\text{Ellipsoid:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \\
\text{Elliptic Paraboloid:} &\quad \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
\text{Hyperbolic Paraboloid:} &\quad \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \\
\text{Cone:} &\quad \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
\text{Hyperboloid of One Sheet:} &\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \\
\text{Hyperboloid of Two Sheets:} &\quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\end{aligned}$$

9.21 Cylindrical Coordinates

To convert from cylindrical to rectangular:

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

To convert from rectangular to cylindrical:

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$$

9.22 Spherical Coordinates

To convert from spherical to rectangular:

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

To convert from rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2 \quad \tan\theta = \frac{y}{x} \quad \cos\phi = \frac{z}{\rho}$$

10 Vector Functions

10.1 Limit of a Vector Function

$$\lim_{t\to a}\vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

10.2 Derivative of a Vector Function

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

10.3 Unit Tangent Vector

$$T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

10.4 Derivative Rules for Vector Functions

$$\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}\,'(t)+\vec{v}\,'(t)$$

$$\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$$

$$\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\ \vec{u}(t) + f(t)\vec{u}\,'(t)$$

$$\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}\,'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}\,'(t)$$

$$\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}\,'(t)\times\vec{v}(t)+\vec{u}(t)\times v\,'(t)$$

$$\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\ \vec{u}\,'(f(t))$$

10.5 Integral of a Vector Function

$$\int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt \right\rangle$$

10.6 Arc Length of a Vector Function

$$L = \int_a^b |r\ '(t)|\ dt$$

10.7 Curvature

$$\kappa = |\frac{d\vec{T}}{ds}| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

10.8 Normal and Binormal Vectors

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

10.9 Velocity and Acceleration

$$\vec{v}(t) = \vec{r}\,'(t)$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

10.10 Parametric Equations of Trajectory

$$x = (v_0 \cos\alpha)t \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

10.11 Tangential and Normal Components of Acceleration

$$\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}$$

10.12 Equations of a Parametric Surface

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

11 Partial Derivatives

11.1 Limit of $f(x, y)$

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

11.2 Strategy to Determine if Limit Exists

1. Substitute in for x and y. If point is defined, limit exists. If not, continue.
2. Approach (x, y) from the x-axis by setting $y = 0$ and taking $\lim_{x\to a}$. Compare this result to approaching (x, y) from the y-axis by setting $x = 0$ and taking $\lim_{y\to a}$. If these results are different, then the limit does not exist. If results are the same, continue.
3. Approach (x, y) from any nonvertical line by setting $y = mx$ and taking $\lim_{x\to a}$. If this limit depends on the value of m, then the limit of the function does not exist. If not, continue.
4. Rewrite the function in cylindrical coordinates and take $\lim_{r\to a}$. If this limit does not exist, then the limit of the function does not exist.

11.3 Continuity

A function is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

11.4 Definition of Partial Derivative

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

11.5 Notation of Partial Derivative

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

11.6 Clairaut's Theorem

If the functions f_{xy} and f_{yx} are both continuous, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

11.7 Tangent Plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

11.8 The Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

11.9 Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

11.10 Gradient

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

11.11 Directional Derivative

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

where $\vec{u} = \langle a, b\rangle$ is a unit vector.

11.12 Maximizing the Directional Derivative

The maximum value of the directional derivative $D_{\vec{u}} f(x)$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(x)$.

11.13 Second Derivative Test

Let $D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
3. If $D < 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a not a local maximum or minimum, but could be a saddle point.

11.14 Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = k$$

2. Evaluate f at all of these points. The largest is the maximum value, and the smallest is the minimum value of f subject to the constraint g.

12 Multiple Integrals

12.1 Volume under a Surface

$$V = \iint\limits_D f(x, y)\ dx\ dy$$

12.2 Average Value of a Function of Two Variables

$$f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x, y)\ dx\ dy$$

12.3 Fubini's Theorem

$$\iint\limits_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dy\ dx = \int_c^d \int_a^b f(x, y)\ dx\ dy$$

12.4 Splitting a Double Integral

$$\iint\limits_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$$

12.5 Double Integral in Polar Coordinates

$$\iint\limits_R f(x, y)\ dA = \int_a^b \int_c^d f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

12.6 Surface Area

$$A(S) = \iint\limits_D |\vec{r}_u \times \vec{r}_v|\ dA$$

where a smooth parametric surface S is given by $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$.

12.7 Surface Area of a Graph

$$A(S) = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

12.8 Triple Integrals in Spherical Coordinates

$$\iiint\limits_E f(x,y,z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

13 Vector Calculus

13.1 Line Integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

13.2 Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

13.3 Path Independence

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D.

13.4 Curl

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}$$

13.5 Conservative Vector Field Test

$\vec{F}$ is conservative if curl $\vec{F} = 0$ and the domain is closed and simply connected.

13.6 Divergence

$$\text{div}(\vec{F}) = \nabla \cdot F$$

13.7 Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_R \text{curl}(\vec{F})\ dx\ dy$$

13.8 Surface Integral

$$\iint\limits_S f(x,y,z)\ dS = \iint\limits_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\ dA$$

13.9 Flux

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dA$$

13.10 Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

13.11 Divergence Theorem

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_E \text{div}(\vec{F})\ dV$$

14 Appendix A: Selected Surface Paramatrizations

14.1 Sphere of Radius ρ

$$\vec{r}(u,v) = \langle \rho \cos u \sin v, \rho \sin u \sin v, \rho \cos v \rangle$$

14.2 Graph of a Function $f(x,y)$

$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

14.3 Graph of a Function $f(\phi, r)$

$$\vec{r}(u,v) = \langle v \cos u, v \sin u, f(u,v) \rangle$$

14.4 Plane Containing $P, \vec{u}$, and $\vec{v}$

$$\vec{r}(s,t) = \vec{OP} + s\vec{u} + t\vec{v}$$

14.5 Surface of Revolution

$$\vec{r}(u,v) = \langle g(v) \cos u, g(v) \sin u, v \rangle$$

where $g(z)$ gives the distance from the z-axis.

14.6 Cylinder

$$\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle$$

14.7 Cone

$$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$$

14.8 Paraboloid

$$\vec{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v\rangle$$

15 Appendix B: Selected Differential Equations

15.1 Heat Equation

$$f_t = f_{xx}$$

15.2 Wave Equation (Wavequation)

$$f_{tt} = f_{xx}$$

15.3 Transport (Advection) Equation

$$f_x = f_t$$

15.4 Laplace Equation

$$f_{xx} = -f_{yy}$$

15.5 Burgers Equation

$$f_{xx} = f_t + ff_x$$