SOLVING SECOND ORDER, HOMOGENEOUS EULER-CAUCHY EQUATIONS: THE CASE OF THE REPEATED ROOT

LANCE DRAGER

In this note, we show how to find the second basic solution for a second order Euler-Cauchy equation in the case of a repeated root of the characteristic equation. We use the method of reduction of order.

Recall that a second order, homogeneous Euler-Cauchy equation has the form

$$x^2 \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \qquad x > 0. \tag{1}$$

The trick for solving this equation is to try for a solution of the form $y = x^m$. Differentiating this function, we have

$$\begin{aligned} y &= x^m \\ y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2}. \end{aligned}$$

Plugging this into (1), we get

$$\begin{aligned} 0 &= x^2 y'' + bxy' + cy \\ &= x^2[m(m-1)x^{m-2}] + bx[mx^{m-1}] + cx^m \\ &= m(m-1)x^m + bmx^m + cx^m \\ &= [m(m-1) + bm + c]x^m. \end{aligned}$$

Thus, $y = x^m$ is a solution if (and only if) m is a root of the characteristic polynomial

$$Q(m) = m(m-1) + bm + c,$$

which simplifies to

$$Q(m) = m^2 + (b-1)m + c. \tag{2}$$

If $Q(m)$ has two distinct roots, either real or complex, we know how to solve the equation. To recall briefly, if $Q(m)$ has two distinct real roots r_1 and r_2, then x^{r_1} and x^{r_2} are two linearly independent solutions of (1), so the general solution of (1) is

$$y = C_1 x^{r_1} + C_2 x^{r_2}.$$

If $Q(m)$ has two distinct complex roots r_1 and r_2, they must be complex conjugates, so we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real.

It's perhaps not so clear what a complex power of x means, but since $x > 0$, we have $x = e^{\ln(x)}$. If the complex power makes any kind of sense, we ought to have

$$\begin{aligned} x^{\alpha+i\beta} &= (e^{\ln(x)})^{\alpha+i\beta} \\ &= e^{\ln(x)(\alpha+i\beta)} \\ &= e^{\alpha\ln(x)+i\beta\ln(x)} \\ &= e^{\alpha\ln(x)}e^{i\beta\ln(x)} \\ &= (e^{\ln(x)})^{\alpha}[\cos(\beta\ln(x)) + i\sin(\beta\ln(x))] \\ &= x^{\alpha}\cos(\beta\ln(x)) + ix^{\alpha}\sin(\beta\ln(x)). \end{aligned}$$

Thus, we *define*

$$x^{\alpha+i\beta} = x^{\alpha}\cos(\beta\ln(x)) + ix^{\alpha}\sin(\beta\ln(x))$$

To justify using this definition to solve Euler-Cauchy equations, we need to check our definition has the property

$$\frac{d}{dx}x^{\alpha+i\beta} = (\alpha+i\beta)x^{(\alpha+i\beta)-1}.$$

We leave this check to the reader. The reader should also check that

$$\overline{x^{\alpha+i\beta}} = x^{\alpha-i\beta}.$$

We can also check that our two solutions x^{r_1} and x^{r_2} are independent. For example, we can use the Wronskian. We conclude that in the case of complex roots the general *complex* solution of (1) is

$$y = A_1x^{r_1} + A_2x^{r_2},$$

where A_1 and A_2 are arbitrary complex constants. To find the real solutions, we set $y = \bar{y}$ and find the conditions that must be satisfied by A_1 and A_2. The result is that in this case the general *real* solution of (1) is

$$y = C_1x^{\alpha}\cos(\beta\ln(x)) + C_2x^{\alpha}\sin(\beta\ln(x))$$

for arbitrary real constants C_1 and C_2.

Finally, there is the case where $Q(m)$ has only one real root r of multiplicity two. In this case, we get one solution $y_1 = x^r$ of (1) and we need to find a second solution that is independent of y_1. This can be done by the method of reduction of order. To apply this method we look for a solution of the form $y = u(x)y_1$, where y_1 is the solution we already know. Before plugging this in, we need to rewrite the differential equation in the right form.

Since r is a root of $Q(m)$ of multiplicity two, we must have $Q(m) = (m-r)^2$. Thus, we have

$$m^2 + (b-1)m + c = Q(m) = (m-r)^2 = m^2 - 2rm + r^2.$$

Thus, we must have

$$\begin{gathered} c = r^2 \\ b - 1 = -2r \implies b = 1 - 2r. \end{gathered}$$

Plugging these values into (1), we can rewire (1) as

$$x^2y'' + (1-2r)xy' + r^2y = 0. \tag{3}$$

Our trial solution is $y = u(x)y_1 = u(x)x^r$. Differentiating this gives

$$\begin{aligned} y &= ux^r \\ y' &= u'x^r + rux^{r-1} \\ y'' &= u''x^r + ru'x^{r-1} + ru'x^{r-1} + r(r-1)ux^{r-2} \\ &= u''x^r + 2ru'x^{r-1} + r(r-1)ux^{r-2}. \end{aligned}$$

Now plug these derivatives into equation (3). Here we go:

$$\begin{aligned} 0 &= x^2y'' + (1-2r)xy' + r^2y \\ &= x^2[u''x^r + 2ru'x^{r-1} + r(r-1)ux^{r-2}] + (1-2r)x[u'x^r + rux^{r-1}] + r^2[ux^r] \\ &= u''x^{r+2} + 2ru'x^{r+1} + r(r-1)ux^r + (1-2r)u'x^{r+1} + r(1-2r)ux^r + r^2ux^r \\ &= u''x^{r+2} + u'[2rx^{r+1} + (1-2r)x^{r+1}] + u[(r(r-1)x^r + r(1-2r)x^r + r^2x^r] \\ &= u''x^{r+2} + u'x^{r+1}[2r + 1 - 2r] + ux^r[r^2 - r + r - 2r^2 + r^2] \\ &= u''x^{r+2} + u'x^{r+1}. \end{aligned}$$

Thus, we get the equation

$$u''x^{r+2} + u'x^{r+1} = 0$$

for u. Dividing both sides by x^{r+2} gives the equation

$$u'' + \frac{1}{x}u' = 0.$$

If we set $v = u'$, we get the first order equation

$$v' + \frac{1}{x}v = 0. \tag{4}$$

(This method is called reduction of order because it reduces a second order problem to a first order problem.)

We can solve (4) by separation of variables, or by noting that it is a first order linear equation with integrating factor x. Multiplying both sides by x gives

$$0 = xv' + v = \frac{d}{dx}(xv),$$

so we have

$$xv = C \implies v = \frac{C}{x}.$$

We're only looking for one u that works, so we can choose the value of C, say $C = 1$. This gives us

$$u' = v = \frac{1}{x}.$$

Integrating, we get that

$$u = \ln(x)$$

(again, we can set the constant of integration to 0 because we're only looking for one u that works). Plugging into our trial solution, $y = ux^r$, we conclude that the second solution of (1) in the case of a double root r is $x^r \ln(x)$. It's pretty clear that these are independent solutions, so the general solution of (1) in the case of a double root r is

$$y = C_1x^r + C_2x^r \ln(x).$$