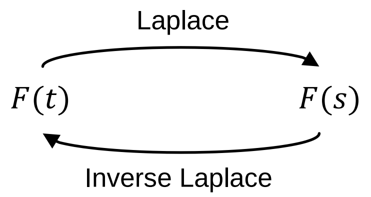
Laplace Summary

Laplace can be used to solve 1st and 2nd order differential equations that are difficult to deal with.

The idea is: Laplace everything,
manipulate it algebraically
inverse Laplace to get the answer

Remember: a function of **t Laplaces** to a function of **s**
and so a function of **s inverse Laplaces** to a function of **t**.



Notation:

When you are using Laplace you could see various notations meaning the same thing – it's good to be familiar with what things mean so that you don't get put off solving the problem.

If you are using Laplace to solve a 1st or 2nd order differential to find x then x is a function of t and could be written as $f(t)$. $\therefore f(0)$ means the same as $x(0)$ etc.

Since x is a function of t (but we don't know what it is) when we Laplace it we could write: $L\{f(t)\}$ or $L\{x\}$ or we could even write $F(s)$ because we know that a function of t Laplaces to a function of s.

Using the Laplace Table:

To Laplace something look at the f(t) column on the table and find which one matches your function. Next decide what your 'n' and/or 'a' and/or 'b' is (whichever ones are necessary) and then Laplace:

Examples:

- $\boldsymbol{L\{cos7t\}}$ **this matches with $\boldsymbol{\cos(at)}$ where $\boldsymbol{a = 7}$**

  $\therefore L\{cos7t\} = \frac{s}{s^2+7^2}$ ← *We replace the a with 7*

- $\boldsymbol{L\{t^3e^{2t}\}}$ **this matches with $\boldsymbol{t^n e^{at}}$ where $\boldsymbol{n = 3, a = 2}$**

  $\therefore L\{t^3e^{2t}\} = \frac{3!}{(s-2)^{3+1}} = \frac{6}{(s-2)^4}$ ← *Replace n with 3 and a with 2*

To inverse Laplace something look at the F(s) column on the table and find which one matches your function. Next decide what your 'n' and/or 'a' and/or 'b' is (whichever ones are necessary) and then inverse Laplace:

Examples:

- $\boldsymbol{L^{-1}\left\{\frac{6}{(s-5)^4}\right\}}$ **this matches with** $\boldsymbol{\frac{n!}{(s-a)^{n+1}}}$ **where** $\boldsymbol{n=3, a=5}$

  $\therefore \boldsymbol{L^{-1}}\left\{\frac{6}{(s-5)^4}\right\} = t^3e^{5t}$ ← *Replace n with 3 and a with 5*

Things to look out for with Laplace:

Look out for multiples: $L\{3t^2\} = L\{3(t^2)\} = 3 \times \frac{2}{s^3} = \frac{6}{s^3}$

$L\{5sin2t\} = 5 \times \frac{2}{s^2+2^2} = \frac{10}{s^2+4}$

Multiply out if necessary: $L\{(t^2-2)^2\} = L\{t^4-4t^2+4\} = \frac{24}{s^5} - \frac{8}{s^3} + \frac{4}{s}$

Things to look out for with Inverse Laplace:

To be able to inverse Laplace something it must look exactly the same as something in the F(s) column of the Laplace table. It can be a multiple or a factor of something in the table but not different. The following are examples of ***patterns*** that appear quite often and the best ways to deal with them in order to be able to inverse Laplace:

Original	Method	Rewrite as
$\frac{8}{s^3}$	Take out the **multiple** so that you have something that looks exactly like a table entry. Notice that 'n' must be 2 (as the power is 3) and so we should have 2! At the top.	$4\left(\frac{2}{s^3}\right)$
$\frac{1}{s^2+9}$	Notice that 9 is 3^2 and so in this case $a=3$ so we should have 3 on the top of the fraction. We have 1 and so we have $\frac{1}{3}$ of what's in the table.	$\frac{1}{3}\left(\frac{3}{s^2+9}\right)$
$\frac{s+2}{s^2-4}$	Remember that both the s and the 2 are divided by $s-4$ and, therefore, can be written **separately**.	$\frac{s}{s^2-4} + \frac{2}{s^2-4}$
$\frac{3s+1}{(s-3)(s+2)}$	**Partial fractions**.	$\frac{2}{(s-3)} + \frac{1}{(s+2)}$
$\frac{s}{s^2+2s+3}$	See if it will **factorise** then use partial fractions. If not – **complete the square**.	$\frac{s}{(s+1)^2+2}$
$\frac{4s-7}{(s-3)^2+25}$	We can see from the table that if there is $s-3$ at the bottom then there has to be $s-3$ at the top as well. So we put the 4 outside a bracket and $s-3$ inside then adjust to make $4s-7$. Notice that $4(s-3)=4s-12$ so we add 5 back on.	$\frac{4(s-3)+5}{(s-3)^2+25}$

If none of the above help then differentiate or integrate to see if it looks like one of the last two results in the Laplace table.

Solving a 2nd Order Differential Equation

Laplace is used to solve differential equations, e.g. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + x = t^3 - 9t^2 + 6t$ where x is a function of t that you need to find.

After transforming the differential equation you need to solve the resulting equation to **make $\boldsymbol{L(x)}$ the subject.** You can then inverse the Laplace transform to find x.

<u>Suggested steps to follow:</u>

Step 1:	Transform everything (using the formula sheet)
Step 2:	Substitute initial conditions in
Step 3:	Keep every term that includes $L(x)$ at the left hand side (LHS) and move everything else to the right hand side (RHS)
Step 4:	Factorise the LHS (taking $L(x)$ out)
Step 5:	Simplify the RHS to one term
Step 6:	Make $L(x)$ the subject
Step 7:	Simplify the RHS
Step 8:	Inverse Laplace to find x

Example:

*Look out for when these are **NOT** 0. Make sure you substitute the correct numbers in.*

Solve $\boldsymbol{\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + x = t^3 - 9t^2 + 6t}$ given the initial conditions, $\boldsymbol{x(0) = 0}$, $\boldsymbol{x'(0) = 0}$

Step 1: $\quad s^2L(x) - x(0) - x'(0) - 3(sL(x) - x(0)) + L(x) = \frac{6}{s^4} - \frac{18}{s^3} + \frac{6}{s^2}$

Step 2: $\quad s^2L(x) - 0 - 0 - 3sL(x) + 3(0) + L(x) = \frac{6}{s^4} - \frac{18}{s^3} + \frac{6}{s^2}$

Step 3: $\quad s^2L(x) - 3sL(x) + L(x) = \frac{6}{s^4} - \frac{18}{s^3} + \frac{6}{s^2}$

Step 4&5: $\quad L(x)(s^2 - 3s + 1) = \frac{6-18s+6s^2}{s^4}$ (tidy up as much as possible)

Step 6: $\quad L(x) = \frac{6-18s+6s^2}{s^4(s^2-3s+1)}$

At this stage see if anything will cancel then you might need to complete the square or factorise the denominator and use partial fractions.

Step 7: $\quad L(x) = \frac{6\cancel{(s^2-3s+1)}}{s^4\cancel{(s^2-3s+1)}} = \frac{6}{s^4}$

Step 8: $\quad L^{-1}\left(\frac{6}{s^4}\right) = t^3$

Therefore the final answer is: $\quad x = t^3$