Limits and Continuity

NOTES

Instructor: Carlos Sotuyo

1.4 Limits (rigorously):

Limit definition: Let $f(x)$ be defined for all x in some open interval containing the number a with the possible exception that $f(x)$ need not be defined at a. We will write:

$$\lim_{x \to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that:

$|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

Use a function's graph to explain the meaning of the previous definition.

Use the definition to prove that $\lim_{x \to 2}(3x - 5) = 1$ and

$\lim_{x \to 3} x^2 = 9$.

1.5. Continuity: Intuitively, the graph of a function can be described as a *continuous curve* if it has no breaks or holes.

$$\lim_{x \to c} = f(c)$$

Provided that the limit at c exists, and the function is defined at c.

Discontinuities: holes, jumps, infinite discontinuities. Use graphs to visualize the difference.

Consequence of continuity: Intermediate value Theorem:

If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.

Use the IVT to prove the existence of at least one zero or root for a continuous polynomial in a given close interval:

Example: prove that $f(x) = x^3 - x - 1$ has a zero in the close interval $[1, 2]$.

1.6: Continuity of trig functions:

Squeezing Theorem:

Let f , g, and h be functions satisfying $g(x) \leq f(x) \leq h(x)$ for all — x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f also has this limit as x approaches c , that is,

$$\lim_{x\to c} f(x) = L$$

Prove:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 1$$

Use the previous result to find:

a) $\lim_{x\to 0} \dfrac{\tan(x)}{x}$

b) $\lim_{x\to 0} \dfrac{\sin(2x)}{x}$

c) $\lim_{x\to 0} \dfrac{\sin(3x)}{\sin(5x)}$