Definite Integral Substitutions. The Logarithm Defined as an Integral

NOTES 23

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5.6 Definite Integral Substitutions. Area between curves:

The Substitution Formula The following formula shows how the limits of integration change when the variable of integration is changed by substitution If $g`$ is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then:

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

Substitute $u = g(x)$, therefore $du = g'(x)dx$, and integrate from $g(a)$ to $g(b)$

Example:

1. Evaluate: $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx$

Solution:

Let $u = x^3 + 1$, $du = 3x^2dx$, $u(-1) = 0$ $u(1) = 2$, therefore:

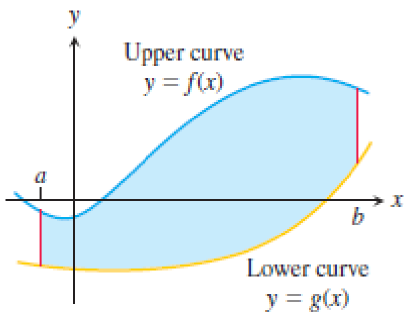
$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx = \int_0^2 \sqrt{u}\ du = \frac{2}{3}u^{3/2}\bigg|_0^2 = \frac{2}{3}\left[2^{3/2}\right] = \frac{4\sqrt{2}}{3}$$

Or, we may integrate with respect to u and change back to x:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx = \int_{-1}^{1} \sqrt{u}\ du = \frac{2}{3}u^{3/2}\bigg|_{-1}^{1} = \frac{2}{3}(x^3+1)^{3/2}\bigg|_{-1}^{1} = \frac{2}{3}\left[2^{3/2}\right] = \frac{4\sqrt{2}}{3}$$

Areas Between Curves:

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$:



DEFINITION: If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$ then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $[f - g]$:

$$A = \int_a^b [f(x) - g(x)]\, dx$$

7.1 The Logarithm Defined as an Integral

Recall the power rule for integrals: $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$

Recall an antiderivative of $\dfrac{1}{x}$:

For $x > 0$, define the natural logarithm function by $\displaystyle \ln x = \int_1^x \frac{1}{t} dt.$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x.

The natural logarithm is the antiderivative of the function $f(u) = 1/u$:

$$\int \frac{1}{u} du = \ln|u| + C.$$