The Fundamental Theorem of Calculus. Indefinite Integrals and the Substitution Method

NOTES 22

Instructor: Carlos Sotuyo

5.4 The Fundamental Theorem of Calculus:

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums.

THEOREM: The Fundamental Theorem of Calculus, Part 1: If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)d(t)$ is continuous on $[a, b]$, and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)d(t) = f(x)$$

Fundamental Theorem, Part 2 (The Evaluation Theorem):

This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

If f is continuous over $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)d(x) = F(b)–F(a)$$

5.5: Indefinite Integrals and the Substitution Method

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)d(x)$ is a number. An indefinite integral $\int f(x)d(x)$ is a function plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as a derivative.

Substitution: Running the Chain Rule Backwards

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx}$$

Applying the concept of the antiderivative,

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C$$

Or,

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

which suggests that the simpler expression du can be substituted for $(du/dx)dx$ when computing an integral. It leads to the substitution method for computing integrals. As with differentials, when computing integrals we have:

$$du = \frac{du}{dx} dx$$

Examples:

1. Find the following indefinite integrals:

a) $\int (x^3 + x)^5 (3x^2 + 1) dx$ b) $\int \sqrt{2x + 1} dx$

THEOREM:

The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then,

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

The Substitution Method:

1. Substitute $u = g(x)$ and $du = (du/dx)dx = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Example: Integrate $\int 2x(x^2 + 5)^7 dx$

Solution: let $u = x^2 + 5$, $\Longrightarrow$ $du = 2x \; dx$ therefore,

$$\int 2x(x^2 + 5)^7 dx = \int u^7 du = \frac{u^8}{8} = \frac{1}{8}(x^2 + 5)^8 + C$$