5.3 The Definite Integral.

NOTES 21

Instructor: Carlos Sotuyo

The Definite Integral:

Riemann Sums The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a Riemann sum, which underlies the theory of the definite integral studied in the next section. If we choose the point ck to be the right-hand endpoint of each subinterval when forming the Riemann sum, it leads to the Riemann sum formula:

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right)$$

becoming an infinite sum of function values $f(x)$ multiplied by *infinitesimal* subinterval widths dx.

In the cases in which the subintervals all have equal width $\Delta x = (b-a)/n$, we can make them thinner by simply increasing their number n. When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

Definition of the Definite Integral:

In the cases where the subintervals all have equal width $\Delta x = (b-a)/n$, we can form each Riemann sum as
$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right)$
Where c_k is chosen in the kth subinterval. When the limit of these Riemann sums as $n \to \infty$ exists and is equal to J, then J is the definite integral of f over $[a,b]$ so

$$J = \int_a^b f(x)d(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right)$$

In short, we define the $\int_a^b f(x)d(x)$ as the limit of the sums $\sum\limits_{k=1}^{n} f(c_k)\Delta x_k$.

Rules satisfied by definite integrals:

1. $\int_a^b f(x)d(x) = -\int_b^a f(x)d(x)$

2. $\int_a^a f(x)d(x) = 0$

3. $\int_a^b kf(x)d(x) = k\int_a^b f(x)d(x)$

4. $\int_a^b [f(x) \pm g(x)]d(x) = \int_a^b f(x)d(x) \pm \int_a^b g(x)d(x)$

5. $\int_a^b f(x)d(x) + \int_b^c f(x)d(x) = \int_a^c f(x)d(x)$

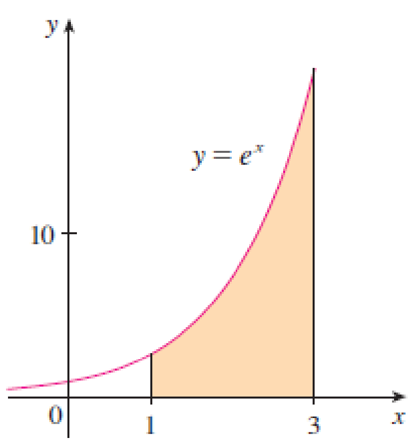
DEFINITION:

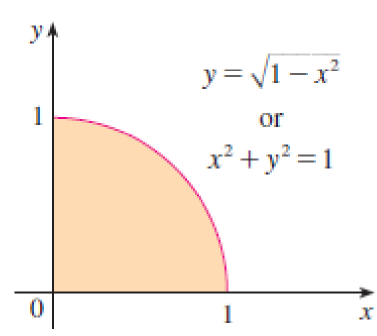
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)d(x)$$

Examples: Set up definite integral to determine area under the curve:

1.





Answers: 1. $\int_1^3 e^x \, d(x)$ 2. $\int_0^1 \sqrt{1-x^2} \, dx$