Area and Estimating with Finite Sums. Sigma Notation and Limits of Finite Sums

NOTES 20

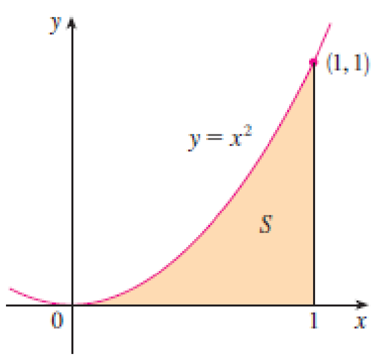
Instructor: Carlos Sotuyo

Area and Estimating with Finite Sums:

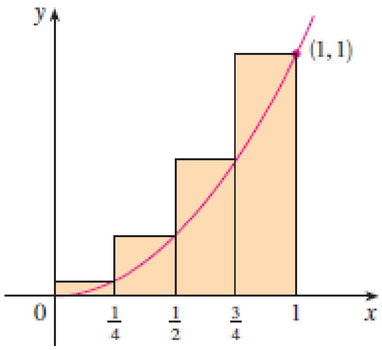
The basis for formulating definite integrals is the construction of appropriate approximations by finite sums.
Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = x^2$, and between the vertical lines $x = 0$ and $x = 1$. Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region S. How, then, can we find the area of S?
While we do not yet have a method for determining the exact area of S, we can approximate it in a simple way.
In each of our computed sums, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (also called length) $\Delta x = (b - a)/n$, and f was evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on. The finite sums then all take the form:

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \ldots + f(c_n)\Delta x.$$

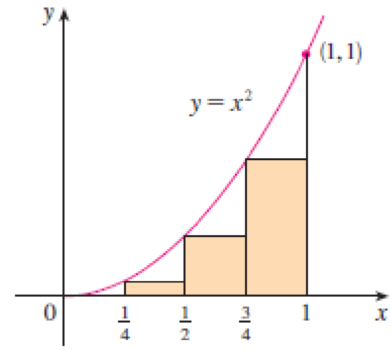
By taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region S.
EXAMPLE: Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



Notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Let's subdivide the interval into four strips.
We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip. In other words, the heights of these rectangles are the values of the function at the left or right endpoints of the subintervals.



Right sum



Left sum

Each rectangle has width $\frac{1}{4}$ and height given by $f(x)$.

A right sum evaluate the function (rectangle's height) at $1/4,\ 2/4,\ 3/4\ \&\ 4/4$ or 1. The rectangle's base is $\Delta x = \frac{b-a}{n} = \frac{1}{4}$.

$R_4 = \sum_{n=1}^{4} \Delta x \cdot f(c_i)$ **Right sum**:

$R_4 = 1/4f(1/4) + 1/4f(2/4) + 1/4f(3/4) + 1/4f(4/4) = 1/4[(1/4)^2 + (2/4)^2 + (3/4)^2 + (4/4)^2] = 0.46875$

Left sum:

$L_4 = 1/4f(0) + 1/4f(1/4) + 1/4f(2/4) + 1/4f(3/4) = 1/4[(0)^2 + (1/4)^2 + (2/4)^2 + (3/4)^2] = 0.21875$

For a midpoint approximation evaluate the function at $1/8,\ 3/8,\ 5/8$ and $7/8$.

What if we divide the interval $[0,1]$ into n subintervals, rectangles, as $n \to \infty$? It leads to:

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \quad \Longrightarrow \quad R_n = \frac{1}{n} \cdot \frac{1}{n^2}\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right)$$

$$R_n = \frac{1}{n^3}\sum_{i=1}^{n} i^2 \qquad R_n = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Taking the limit as $n \to \infty$, yields:

$$\lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(2n^2+3n+1)}{6n^2} = \frac{1}{3}$$ which is the integral of $y = x^2$ from 0 to 1.

We can establish that: $\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$

If is integrable on $[a,b]$ then the limit exists and gives the same value no matter how we choose the sample points x_i. To simplify the calculation of the integral we often take the sample points to be right endpoints. Definition of an integral simplifies as follows:

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$