Rates of Change

NOTES 01

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2.1: Rates of Change and Tangent Lines to Curves:

Everything changes over time; in mathematics, we are specially interested in describing how quantities change over time. Quantities may be a function of time; for instance, the motion of a particle.
For a function describing position with respect to time, $s(t)$, the rate of change in a time interval $[t_1, t_2]$ is given by:

$$\text{Average rate of change} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}$$

Of course, this is how we learn to calculate the slope of a line in algebra. In general for a function, $f(x)$,

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

In short, the *average rate of change* between two points (x_1, y_1) and (x_2, y_2) is given by the **slope** of the secant line connecting these two points.

$$\text{Average rate of change} = m = \frac{\Delta y}{\Delta x}$$

Let's apply this concept to a free falling object. The following paragraph, was taken from Thomas' Calculus, 13th ed.:

Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called free fall. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in feet after t seconds, then Galileo's law is $y = 16t^2$.

Example 1:
Using Galileo's law, $S(t) = 16t^2$ (feet per seconds) and considering that a moving object's average speed during an interval of time is found by dividing the distance covered by the time elapsed. A rock breaks loose from the top of a tall cliff. What is its average speed:
(a) during the first 2 sec of fall? [from $t_1 = 0$ to $t_2 = 2$]
(b) during the 1-sec interval between second 1 and second 2?
Answer: a) $\text{Average rate of change} = \text{Average speed} = \frac{\Delta S}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\ ft/s$

b) $\text{Average rate of change} = \text{Average speed} = \frac{\Delta S}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\ ft/s$

Example 2: What would be the instantaneous speed of the object at $t = 1$?

Speed implies change in position. Does position change in an instant? The time elapsed in an instant is very very small. We say that the difference between t_1 and t_2 *tends to zero*. Instead of t let's use a more general independent variable, x. If h represent a very small number, a number that in calculus we say *approaches zero*, then $x_2 = x_1 + h$ so we can show that the formula for **Average rate of change** becomes:

$$\frac{f(x_1 + h) - f(x_1)}{h}$$

as h approaches zero [notation: $x \to 0$]

In our algebra courses we called this formula the *difference quotient*, let's called instantaneous rate of change:

$$\text{Instantaneous rate of change} = \frac{f(x_1 + h) - f(x_1)}{h}$$

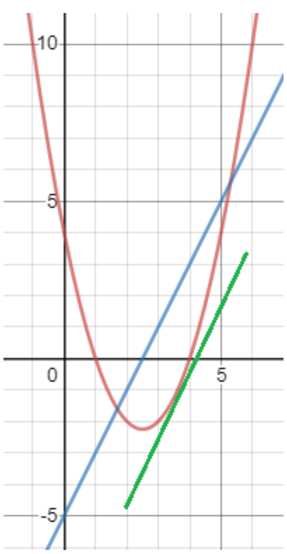
as $h \to 0$, but $h \neq 0$

$$\text{Instantaneous speed} = \frac{S(t + h) - S(t)}{h} = \frac{16(t + h)^2 - 16t^2}{h}$$

which simplifies to:

$= \dfrac{16(t^2 + 2th + h^2) - 16t^2}{h} = \dfrac{32th + 16h^2}{h} = 32t + h.$ Since $h \to 0$, the Instantaneous speed is given by $32t$, at $t = 1$, Speed $= 32ft/s$

Example 3: Relate the average of change to the secant line to a curve, and the instantaneous rate of change to the tangent line to a point. Notice that the limiting position of the secant line is called the tangent line to the curve at a point.



From the slope formula for a line, given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we can write the general formula for a line that passes trough a point (x_1, y_1) where x and y represent all others points on the line as:

$$y - y_1 = m(x - x_1)$$