Concavity and Curve Sketching

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Instructor: Carlos Sotuyo

4.4 Concavity and Curve Sketching:

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. With the knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior, we can draw an accurate graph of a function.

The Second Derivative Test for Concavity:

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1:

(a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

Example 2: Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

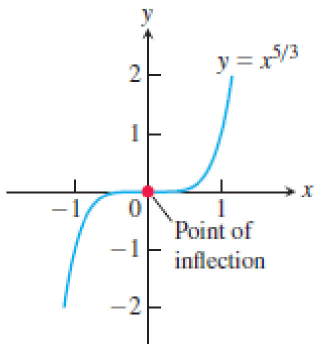
Solution: The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive.

Points of Inflection:

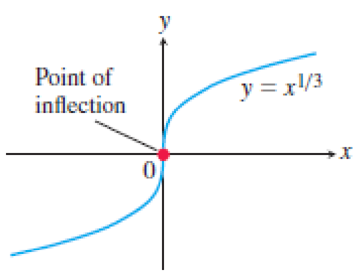
Definition: A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

At a point of inflection $(c, f(c)$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$.



$y = x^{1/3}$ A point of inflection where y' and y'' fail to exist:



Second Derivative Test for Local Extrema:

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example: Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$ using the following steps:

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.