Extreme Values of Functions. The Mean Value Theorem.

NOTES 15

Instructor: Carlos Sotuyo

4.1 Extreme Values of Functions:

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative.

Definitions: Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if:

$f(x) \leq f(c)$ $\quad\quad$ $\forall\ x\ in$ D $\quad\quad$ where $\forall$ is read *for all*

and an absolute mínimum value D at c if

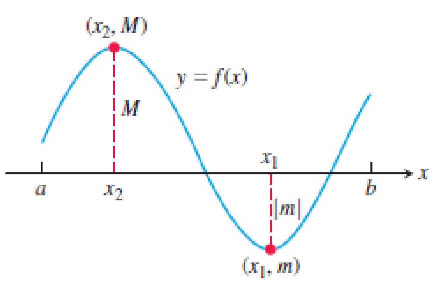
$f(x) \geq f(c)$ $\quad\quad$ $\forall\ x\ in$ D

Maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima.

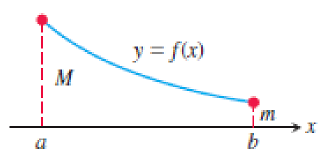
The Extreme Value Theorem:

> If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

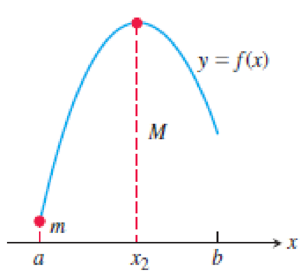
Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$ are illustrated below:



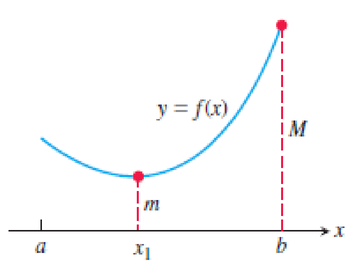
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



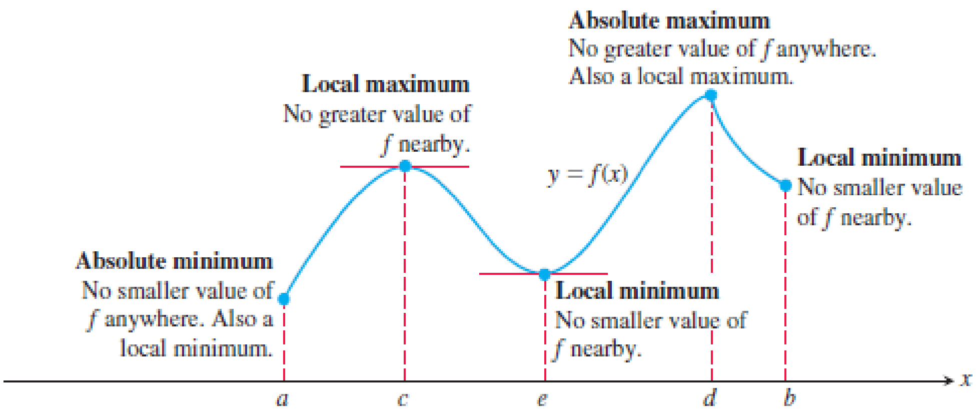
Minimum at interior point, maximum at endpoint

Definitions:

A function f has a local maximum value at a point c within its domain D if $f(x) \leq f(c)$ $\forall x \in D$ lying in some open interval containing c.

A function f has a local minimum value at a point c within its domain D if $f(x) \geq f(c)$ $\forall x \in D$ lying in some open interval containing c.

Identifying types of maxima and minima for a function with domain $[a, b]$:



Finding Extrema:

The First Derivative Theorem for Local Extreme Values:

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Definition:

An interior point of the domain of a function f where $f' = 0$ or undefined is a critical point of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval:

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example 1:

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

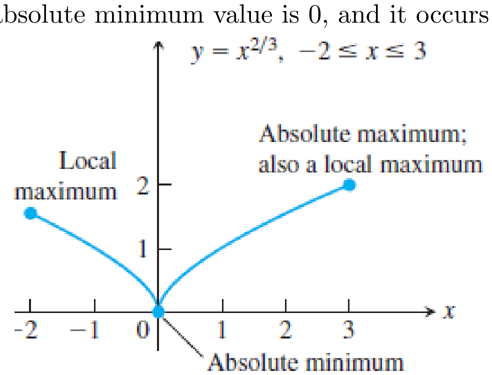
Solution:

$f'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$ A critical point is a point at which the first derivative is equal to zero or undefined.

In this case $f'(x)$ is undefined at $x = 0$ and $f(0) = 0$

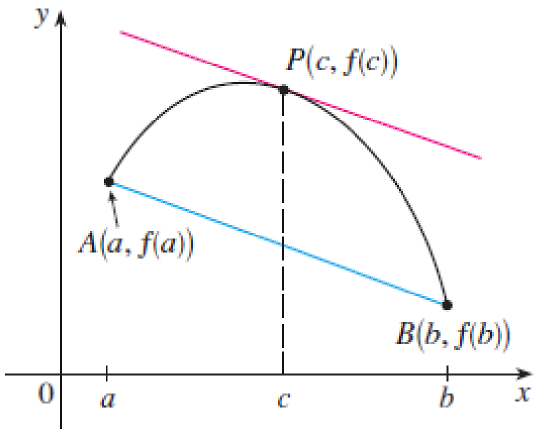
We also need to evaluate the function at the endpoints, namely, -2 and 3 : $f(-2) = \sqrt[3]{4} \approx 1.587$, $f(3) = \sqrt[3]{9} \approx 2.080$

We can see that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.080$ and it occurs at the right endpoint, $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp.



4.2 The Mean Value Theorem:

Examine the graph:



The Mean Value Theorem:

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Example 2:

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value c $\in$ $(0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value c $\in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by $\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}$.

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that $\frac{1}{2\sqrt{c}} = \frac{1}{3}$.

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

Example 2 taken from https://opentextbc.ca/