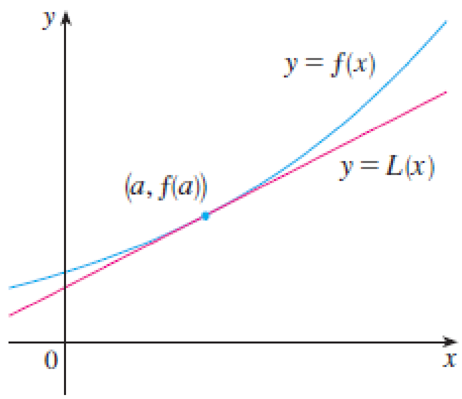
Linearization and Differentials

NOTES 14

Instructor: Carlos Sotuyo

3.11 Linearization:



Notice that a curve lies very close to its tangent line near the point of tangency. We may use the tangent line at $(a, f(a))$ as an approximation to the curve when x is near a. An equation of this tangent line is:

$$y = f(a) + f'(a)(x - a)$$

This result is obtained from the point-slope formula for a line, substituting the slope, m, by $f'(a)$. Therefore, the linear approximation or tangent line approximation of f at a is given by:

$$L(x) \approx f(a) + f'(a)(x - a)$$

Examples:

1. Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$. Use this result to evaluate $\sqrt{1.005}$

$f(0) = 1$ $\qquad f'(x) = \frac{1}{2}(1 + x)^{-1/2}$ $\qquad f'(0) = \frac{1}{2}$

$L(x) = f(0) + f'(0)(x - 0)$ $\qquad L(x) = 1 + \frac{1}{2}x$

$\sqrt{1.005} \approx 1 + \frac{1}{2}(0.005) \approx 1.00250$

2. Find the linearization of $f(x) = \cos x$ at $x = \dfrac{\pi}{2}$.

$f\left(\dfrac{\pi}{2}\right) = 0$ $\quad f'(x) = -\sin x$ $\quad f'\left(\dfrac{\pi}{2}\right) = -1$

$L(x) = f\left(\dfrac{\pi}{2}\right) + f'\left(\dfrac{\pi}{2}\right)\left(x - \dfrac{\pi}{2}\right)$ $\quad L(x) = 0 + -1\left(x - \dfrac{\pi}{2}\right)$ $\quad \therefore\ L(x) = -x + \dfrac{\pi}{2}$

3. Show that the linearization of $f(x) = (1+x)^k$ near zero is $L(x) = 1 + kx$

$f(0) = 1$ $\quad f'(x) = k(1+x)^{k-1}$ $\quad \therefore\ f'(0) = k$

$L(x) = 1 + k(x-0)$ $\quad \Longrightarrow \quad$ $L(x) = 1 + kx$

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

Definition:

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is:

$$dy = f'(x)dx$$

Example 4:

a. Find dy if $y = x^5 + 37x$

b. Find the value of dy when $x = 1$ and $dx = 0.2$

Answer:

a. $dy = (5x^4 + 37)\ dx$

b. $dy = (5 \cdot 1^4 + 37)(0.2) = 8.4$

Estimating with Differentials:

Example 5:

The radius r of a circle increases from 10 m to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Answer:

$dA = 2\pi r\ dr = 2\pi(10)(0.1) = 2\pi\ \text{m}^2$

$A(10 + 0.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi\ \text{m}^2$

Finding the area directly, no approximation involved:

$A(10.1) = \pi(10.1)^2 = 102.01\pi\ \text{m}^2$

The error of the estimate is $102.01\pi - 102\pi = 0.01\pi\ \text{m}^2$