Related rates

NOTES 13

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Related rates problems:

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time. (Taken from Calculus, 6th Ed, by J. Stewart)
The following example involves relating rates of change that occur with respect to time. It is possible to relate rates of change that occur with respect to a quantity other than time.

Discuss:

1. Show that the rate at which the area of a circle changes, when the radius changes by an infinitesimal amount is given by the change in the circle's circumference.
2. Show that the rate at which the volume of a sphere changes, when the radius changes by an infinitesimal amount is given by the change in the sphere's surface area.

Strategy solving related rates problems:

* Express the given information and the required rate in terms of derivatives.
* Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
* Use the Chain Rule and implicit differentiation to differentiate both sides of the equation with respect to time.
* Substitute the given information into the resulting equation and solve for the unknown rate.

Examples:

1. Area Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.
2. Assume that $2x^2 + 3y = 11$ and $dy/dt = -2$. Find dx/dt when $x = 1$.
3. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
4. If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3\ m$ at what rate does the cube's volume change?

5. The radius r and height h of a right circular cylinder are related to the cylinder's volume by the formula $V = \pi r^2 h$.

a. How is dV/dt related to dh/dt if r is constant?

b. How is dV/dt related to dr/dt if h is constant?

c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

6. The voltage V (Volts), current I (Amperes), and resistance R (Ohms) of an electric circuit are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 Amp/sec.

a. What is the value of dV/dt?

b. What is the value of dI/dt?

c. What equation relates dR/dt to dV/dt and dI/dt?

d. Find the rate at which R is changing when V = 12 volts and I = 2 Amps. Is R increasing, or decreasing?

Solutions:

1. Area Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to to dr/dt.

Answer: $dA/dt = 2\pi r \; dr/dt$

2. Assume that $2x^2 + 3y = 11$ and $dy/dt = -2$. Find dx/dt when $x = 1$.

Answer:

$4x \; dx/dt + 3 \; dy/dt = 0 \qquad 4(1) \; dx/dt + 3(-2) = 0 \qquad \Longrightarrow \qquad dx/dt = \dfrac{3}{2}$

3. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?

Answer:

$x^2y^3 = 4/27$ for $x = 2 \;\; 4y^3 = 4/27 \;\; \therefore \;\; y = 1/3$

Taking the derivative:

$2xy^3 \; dx/dt + 3x^2y^2 \; dy/dt = 0$ substituting values: $2(2)(1/3)^3 \; dx/dt + 3(4)(1/3)^2(1/2) = 0$ which yields $dx/dt = -9/2$

4. If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's volume change?

Answer:

$V = x^3 \qquad \therefore \qquad dV/dt = 3x^2 \; dx/dt \qquad dV/dt = 3(3 \; m)^2 \; 5 \; m/min = -135 \; m^3/min$

5. The radius r and height h of a right circular cylinder are related to the cylinder's volume by the formula $V = \pi r^2 h$.

a. How is dV/dt related to dh/dt if r is constant?

b. How is dV/dt related to dr/dt if h is constant?

c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

Answer to number 5:

a) $dV/dt = \pi r^2 \; dh/dt$

b) $dV/dt = 2\pi rh \; dr/dt$

c) $dV/dt = 2\pi rh \; dr/dt + \pi r^2 \; dh/dt$

6. The voltage V (Volts), current I (Amperes), and resistance R (Ohms) of an electric circuit are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 Amp/sec.

a. What is the value of dV/dt?

b. What is the value of dI/dt?

c. What equation relates dR/dt to dV/dt and dI/dt?

d. Find the rate at which R is changing when V = 12 volts and I = 2 Amps. Is R increasing, or decreasing?

Answer:

a) $dV/dt = 1$ Volt/sec

b) $dI/dt = -1/3$ Amp/sec

c) $V = IR \qquad \therefore \qquad R = \dfrac{V}{I}$ differentiate R with respect to time using the quotient rule:

$$dR/dt = \frac{I \; dV/dt - V \; dI/dt}{I^2} = \frac{1}{I}\frac{dV}{dt} - \frac{V}{I^2}\frac{dI}{dt} = \frac{1}{I}\left[\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right] = \frac{1}{2}\left(1 - \frac{12}{2} \cdot -\frac{1}{3}\right) = 3.5$$ Ohms/secs; R is increasing since $dR/dt > 0$.