Derivatives of Inverse Trigonometric Functions

NOTES 12

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Derivatives of Inverse Trigonometric Functions: $\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x,$

Considering that, if f is a differentiable and one-to-one function, then $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$

For $f(x) = \sin x \quad f^{-1}(x) = \sin^{-1} x$ and $f'(x) = \cos x$

Then,

$\left(f^{-1}\right)'(x) = \dfrac{1}{\cos(\sin^{-1} x)}$

Recall that $\sin^{-1} x$ is the angle, say θ, whose sine is x, in the interval of definition of inverse sine, $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. Therefore, we may write: $\left(f^{-1}\right)'(x) = \dfrac{1}{\cos(\theta)}$

And since $\sin(\theta) = x$, by the Pythagorean trigonometric identity, $\sin^2(\theta) + \cos^2(\theta) = 1 \implies x^2 + \cos^2(\theta) = 1$. Solving for $\cos(\theta) = \sqrt{1 - x^2}$ we obtain the following result for the derivative of the inverse sine function:

$$\left(f^{-1}\right)'(x) = \frac{1}{\sqrt{1 - x^2}}$$

By differentiating implicitly $y = \sin^{-1} x$; equivalent to $\sin y = x$ we may also show that the derivative of the inverse sine function is $y' = \dfrac{1}{\sqrt{1 - x^2}}$. Notice that implicit differentiation of $\sin y = x$ yields $\cos y \cdot y' = 1$ or $y' = \dfrac{1}{\cos y}$, and, writing the trig identity as follows: $\sin^2(y) + \cos^2(y) = 1$ we obtain again the same result for $y = \sin^{-1} x$,

$$y' = \frac{1}{\sqrt{1 - x^2}}$$

Let's use the implicit differentiation procedure to show that:

For $y = \cos^{-1}(x)$, the derivative is $y' = -\dfrac{1}{\sqrt{1 - x^2}}$

For $y = \tan^{-1}(x)$, the derivative is $y' = \dfrac{1}{1 + x^2}$

For $y = \cot^{-1}(x)$, the derivative is $y' = -\dfrac{1}{1 + x^2}$

For $y = \sec^{-1}(x)$, the derivative is $y' = \dfrac{1}{|x|\sqrt{x^2 - 1}}$

For $y = \csc^{-1}(x)$, the derivative is $y' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

Proofs on next page:

i. Derivative of $y = \cos^{-1}(x)$: $\implies$ $\cos y = x$; which, by implicit differentiation yields:

$-\sin y \cdot y' = 1 \qquad \implies \qquad y' = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} \qquad \implies \qquad y' = -\dfrac{1}{\sqrt{1-x^2}}$

ii. Derivative of $y = \tan^{-1}(x)$: $\implies$ $\tan y = x$; which, by implicit differentiation yields:

$\sec^2 y \cdot y' = 1 \qquad \implies \qquad y' = \dfrac{1}{\sec^2 y}$ By the identity, $1+\tan^2 y = \sec^2 y$ $\qquad \therefore \qquad y' = \dfrac{1}{1+\tan^2 y}$

OR, $\quad y' = \dfrac{1}{1+x^2}$.

iii. Derivative of $y = \cot^{-1}(x)$: $\implies$ $\cot y = x$; which, by implicit differentiation yields:

$-\csc^2 y \cdot y' = 1 \qquad \implies \qquad y' = -\dfrac{1}{\csc^2 y}$ By the identity, $1+\cot^2 y = \csc^2 y$ $\qquad \therefore \qquad y' = \dfrac{1}{1+\cot^2 y}$

OR, $\quad y' = -\dfrac{1}{1+x^2}$.

iv. Derivative of $y = \sec^{-1}(x)$: $\implies$ $\sec y = x$; which, by implicit differentiation yields:

$\sec y \tan y \cdot y' = 1 \qquad \implies \qquad y' = \dfrac{1}{\sec y \tan y}$ By the identity, $1+\tan^2 y = \sec^2 y$ $\qquad \therefore \qquad y' = \dfrac{1}{|x|\sqrt{x^2-1}}$

v. Derivative of $y = \csc^{-1}(x)$: $\implies$ $\csc y = x$; which, by implicit differentiation yields:

$-\csc y \cot y \cdot y' = 1 \qquad \implies \qquad y' = -\dfrac{1}{\csc y \cot y}$ By the identity, $1+\cot^2 y = \csc^2 y$ $\qquad \therefore \qquad y' = -\dfrac{1}{|x|\sqrt{x^2-1}}$

Chain rule and Derivatives of Inverse Trigonometric Functions:

Notice that, if instead of x we have a function of x say, $u(x)$ then, applying the chain rule, we obtain:

For $y = \sin^{-1}(u)$, the derivative is $y' = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

For $y = \cos^{-1}(u)$, the derivative is $y' = -\dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

For $y = \tan^{-1}(u)$, the derivative is $y' = \dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

For $y = \cot^{-1}(u)$, the derivative is $y' = -\dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

For $y = \sec^{-1}(u)$, the derivative is $y' = \dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$

For $y = \csc^{-1}(u)$, the derivative is $y' = -\dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$