Derivatives of Inverse Functions and Logarithms

NOTES 11

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Inverse functions:

An inverse function is a function that undoes the action of the another function. A function g is the inverse of a function f if whenever $y = f(x)$ then $x = g(y)$. The inverse of $f(x)$ is denoted $f^{-1}(x)$. The composition of a functions and its inverse is the input, x:

$$f\left(f^{-1}(x)\right) = x$$

Therefore, taking he derivative to the previous equation (chain rule), we obtain:

$$\left[f\left(f^{-1}(x)\right)\right]' = f'\left(f^{-1}(x)\right) \cdot \left(f^{-1}\right)'(x) = 1$$

Now, solving for the derivative of the inverse function:

Derivative of the inverse function

If f is a differentiable and one-to-one function, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Based on the definition of an inverse function, if $f(a) = b$ then $f^{-1}(b) = a$. Therefore, applying the previous result for the derivative of an inverse function:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)} = \frac{1}{f'(a)}$$

Example 1: Given $f(x) = \frac{1}{4}x^3 + x - 1$ find $\left(f^{-1}\right)'(3)$

Since, by inspection, $f(2) = 3$ then $\left(f^{-1}\right)(3) = 2$.

Therefore:

$\left(f^{-1}\right)'(3) = \frac{1}{f'\left(f^{-1}(3)\right)} = \frac{1}{f'(2)}$ obtain $f'(x) = \frac{3}{4}x^2 + 1$ evaluated at $x = 3$ yields $f'(2) = 4 \implies \left(f^{-1}\right)'(3) = \frac{1}{4}$.

Finding derivatives of logarithmic and exponential functions:

i. For $f(x) = e^x$ use the relationship between the inverse function derivative and the function's derivative to show that the derivative of $f(x) = \ln x$ is $\frac{1}{x}$.

$f(x) = e^x \implies f^{-1}(x) = \ln x$ Also, $f'(x) = e^x$

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} \implies \left(f^{-1}\right)'(x) = \frac{1}{e^{(\ln x)}} = \frac{1}{x}$$

ii. Use the previous result to show that for $f(x) = \log_a x$ the derivative is $f'(x) = \frac{1}{x \ln a}$

In algebra we learn that $\log_a x = \frac{\ln x}{\ln a}$; therefore, the derivative of $f(x) = \log_a x$ is $f'(x) = \frac{1}{x \ln a}$

iii. Show that the derivative of $f(x) = a^x$ is $f(x) = a^x \ln a$. Hint, $a^x = \left(e^{\ln a}\right)^x$

$f(x) = a^x = e^{x \ln a}$; therefore, since derivative of $f(x) = e^{u(x)}$ is $f'(x) = e^{u(x)} \cdot u'(x)$, then, the derivative of $f(x) = e^{x \ln a}$ is $f'(x) = e^{x \ln a} \cdot (x \ln a)' = e^{x \ln a} \cdot \ln a = a^x \ln a$

Logarithmic differentiation:

Taking the derivatives of some functions can be simplified by using natural logarithm. Take natural log of both sides of the equation, and differentiate implicitly. This is called logarithmic differentiation. An example:

i. Find the derivative of $f(x) = x^x$

Write $y = x^x$ take *ln* both sides of the equation:

$\ln y = \ln x^x$ which is equivalent to $\ln y = x \ln x$

Taking the derivative:

$\frac{1}{y} y' = \ln x + x \cdot \frac{1}{x} \implies y' = y\,(\ln x + 1)$ and substituting y by the original function $y = x^x$, yields:

$y' = x^x\,(\ln x + 1))$

ii. Use logarithmic differentiation to show that the derivative of $y = x^n$ is $y' = nx^{n-1}$ for all real numbers n.

$$\ln y = \ln x^n \implies \ln y = n \ln x \implies \frac{1}{y} y' = \frac{n}{x} \implies y' = y \cdot \frac{n}{x} = x^n \cdot \frac{n}{x} \implies y' = nx^{n-1}$$

Example 2: Knowing that the derivative of $f(x) = \ln x$ is $f'(x) = \frac{1}{x}$ and using the limit definition of the derivative, show that $e = \lim_{x \to 0} (1 + x)^{1/x}$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \implies f'(1) = \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} \implies f'(1) = \lim_{h \to 0} \frac{1}{h} \left[\ln(1+h)\right]$$

$$\implies f'(1) = \lim_{h \to 0} \left[\ln(1+h)\right]^{1/h} \implies \ln\left[\lim_{h \to 0} (1+h)^{1/h}\right] = 1 \implies \lim_{h \to 0} (1+h)^{1/h} = e.$$

Definition of e as a limit:

$$\lim_{x \to 0} (1+x)^{1/x} = e \quad \text{which is equivalent to} \quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$