Implicit Differentiation

NOTES 10

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Differentiate the following implicit functions, in which y is not explicitly stated as a function of x:

a) $xy + x = 1$

b) $x^3 + y^3 - 9xy = 0$

The first function is equivalent to $y = \frac{1-x}{x} = \frac{1}{x} - 1$; and the derivative of y with respect to x is $\frac{dy}{dx} = -\frac{1}{x^2}$

What about the second function, known as the *folium of Descartes*? It is difficult to solve for y explicitly as a function of x.

How do we **differentiate an explicitly defined function**?

We do know how to differentiate the term x^3. Yes, but let's write the notation in details:

$$\frac{d}{dx}\left(x^3\right) = 3x^2\frac{dx}{dx} = 3x^2$$

Now, in order to differentiate y^3 with respect to x we differentiate y^3 with respect to y and then multiply by $\frac{dy}{dx}$ as follows:

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{dx}y^3 \cdot \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

In summary: to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by dy/dx.

Implicit differentiation:

The implicit differentiation of y with respect to x:

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dx}\left(f(y)\right) \cdot \frac{dy}{dx}$$

Let's complete the differentiation of b) $x^3 + y^3 - 9xy = 0$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0) \qquad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\left(y\frac{d}{dx}x + x\frac{d}{dx}y\right) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9(y \cdot 1 + x \cdot \frac{dy}{dx}) = 0 \qquad 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2 \qquad \Longrightarrow \quad \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} \qquad \Longrightarrow \quad \frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

For simplicity, instead of writing $\frac{dy}{dx}$ we may write y'. A few more examples:

Examples:

i. Find $\frac{dy}{dx}$ for explicitly defined function: $2x^3 - 3y^2 = 8$

$$6x^2 - 6yy' = 0 \qquad \Longrightarrow \quad -6yy' = -6x^2 \qquad \Longrightarrow \quad y' = \frac{x^2}{y}$$

ii. Find y'' explicitly defined function $2x^3 - 3y^2 = 8$

Take the derivative of the first derivative, $y' = \frac{x^2}{y}$ as follows:

$$y'' = \frac{2xy - x^2y'}{y^2} \qquad \Longrightarrow \quad y'' = \frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2} \qquad \Longrightarrow \quad y'' = \frac{2xy^2 - x^4}{y^3}$$

iii. Find $\frac{dy}{dx}$ for explicitly defined function: $y^2 = x^2 + \sin(xy)$

$$2yy' = 2x + \cos(xy)\left(1 \cdot y + xy'\right) \qquad \Longrightarrow \quad 2yy' = 2x + y\cos(xy) + \big(x\cos(xy)\big)\,y'$$

$$2yy' - \big(x\cos(xy)\big)\,y' = 2x + y\cos(xy) \qquad \Longrightarrow \quad y'\,\big(2y - x\cos(xy)\big) = 2x + y\cos(xy)$$

$$\Longrightarrow \quad y' = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

iv. Find the equation of the normal line to the circle defined by $x^2 + y^2 = 4$ at $\left(1, \sqrt{3}\right)$.

The normal line is perpendicular to the tangent line; their slopes are negative reciprocal of each other. Find the slope of the tangent line first. It is given by $m = f'(1, \sqrt{3})$. Implicit differentiation of the circle equation yields:

$$2x + 2yy' = 0 \qquad \Longrightarrow \quad y' = -\frac{x}{y}$$

Therefore, slope of tangent line at $\left(1, \sqrt{3}\right)$ is $m = -\frac{1}{\sqrt{3}}$. The slope of the normal line is $\sqrt{3}$.

Using the point-slope formula: $y - y_1 = m(x - x_1)$ and substituting:

$y - \sqrt{3} = \sqrt{3}\,(x - 1) \quad \Longrightarrow \quad y - \sqrt{3} = \sqrt{3}x - \sqrt{3}$ Therefore the equation of the normal line at $\left(1, \sqrt{3}\right)$ is $y = \sqrt{3}x$