Chain Rule

NOTES 09

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Derivative of a Composite Function:

Example: The function $y = (x^2 - 1)^3$ is the composite of the functions $y = u^3$ and $u = x^2 - 1$. Instead of finding the derivative of y with respect to x directly, we may do it in two steps: find the derivative of y with respect to u, $\dfrac{dy}{du}$, the derivative of u with respect to x, $\dfrac{du}{dx}$, and combine the two:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

For $y = (x^2 - 1)^3$: $\quad y = u^3 \quad u = x^2 - 1 \quad \dfrac{dy}{du} = 3u^2 \quad \dfrac{du}{dx} = 2x$

Substituting into: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$, yields: $\dfrac{dy}{dx} = 3u^2 \cdot 2x$, and since $u = x^2 - 1$, $\dfrac{dy}{dx} = 6x(x^2 - 1)^2$

The Chain Rule:

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)(x) = f(g(x)) = f'(g(x)) \cdot g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Outside-Inside Rule.

If $y = f(g(x))$, then $\dfrac{dy}{dx} = f'(g(x))g'(x)$. In words: differentiate the *outside* function f and evaluate it at the *inside* function $g(x)$ left alone; then multiply by the derivative of the *inside* function.

Examples to be discussed in class:

Differentiate:

1) $y = \sin(x^2 + x)$

$$\frac{d}{dx}\sin(x^2 + x) = [\cos(x^2 + x)] \cdot (2x + 1) = (2x + 1)\cos(x^2 + x)$$

2) $y = \left(5x^3 - x^4\right)^7$

$$\frac{d}{dx}\left(5x^3 - x^4\right)^7 = 7\left(5x^3 - x^4\right)^6 (15x^2 - 4x^3)$$

3) $y = \dfrac{1}{3x - 2} = (3x - 2)^{-1}$

$$\frac{d}{dx}(3x - 2)^{-1} = -1\,(3x - 2)^{-2} \cdot 3 = -\frac{3}{(3x - 2)^2}$$

4) $y = \sin^5 x$

$y = \sin^5 x$ is equivalent to $y = (\sin x)^5$

$$\frac{d}{dx}(\sin x)^5 = 5\,(\sin x)^4 \cos x = 5 \sin^4 x \cos x$$