Derivatives of Trigonometric Functions

NOTES 08

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How to differentiate the six basic trigonometric functions? Let's begin with the sine function.

Derivative of the Sine Function:

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits with the angle sum identity for the sine function: $\sin(x + h) = \sin x \cos h + \cos x \sin h$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \implies \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \implies \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$\lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \left(\cos x \cdot \frac{\sin h}{h}\right) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

Definition:

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x$$

In short: the rate at which the sine function changes is given by the cosine function. Notice, for example, that at $\theta = \pi/2$ the slope of the tangent line to the sine function is zero, and $\cos(\pi/2) = 0$

Derivative of the Cosine Function:

Using the trigonometric identity $\cos(x+h) = \cos x \cos h - \sin x \sin h$, and the definition of the derivative, $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ it can be shown that:

Definition:

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivatives of the Other Basic Trigonometric Functions:

Finding the derivative of the **tangent** function:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivative of the **cotangent** function:

It can be shown that: $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x$

Derivative of the **secant** function:

It can be shown that: $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$

Derivative of the **cosecant** function:

It can be shown that: $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$

Derivatives of Trig functions:

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$