Tangent Lines and the Derivative at a Point. The Derivative as a Function

NOTES 06

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Rates of Change: Derivative at a Point

The expression:

$$\frac{f(x_0+h)-f(x_0)}{h}, \quad h \neq 0$$

Is called the *difference quotient of f at* x_0 *with increment* h. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation:

Definition:

The **derivative of a function f at a point** x_0 denoted $f'(x_0)$ is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

Provided this limit exists.

The following are all interpretations for the limit of the difference quotient:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$.
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at at $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

An alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity:

$$f'(c) = \lim_{c\to x} \frac{f(c)-f(x)}{c-x}$$

Example 1: Use the limit definition of the derivative to find $f'(x)$:

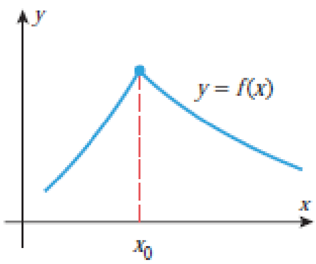
a) $y = 4 - x^2$,

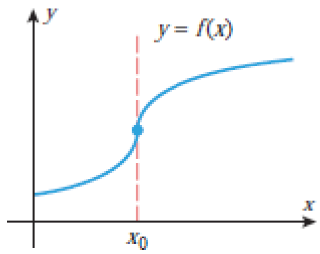
b) $y = \sqrt{x}$,

c) $y = \dfrac{3}{x}$.

Answers to be discussed in class.

According to Definition the derivative exists precisely when the limit exists. That limit does not exist when the curve $y = f(x)$ does not have a tangent line at $x = c$ or when the curve does have a tangent line, but the tangent line has infinite slope. f is not differentiable at any point x_0 where the secant lines from $P(x_0, f(x_0))$ to points $Q(x, f(x))$ distinct from P do not approach a unique nonvertical limiting position as $x \to x_0$





Example 2: Discuss that both, $y = |x|$ and $y = x^{2/3}$ are not differentiable at the origin.

Conclude that, while differentiability implies continuity; continuity does not implies differentiability.

ANSWERS:

Example 1:

a) $f'(x) = \lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}, \implies \lim_{h \to 0} \dfrac{(4 - (x+h)^2 - (4 - x^2))}{h} \implies \lim_{h \to 0} \dfrac{(4 - x^2 - 2xh - h^2 - 4 + x^2)}{h}$

$\implies \lim_{h \to 0} \dfrac{(-2xh - h^2)}{h} \implies \lim_{h \to 0} -2x - h = -2x$

b) $f'(x) = \lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}, \implies \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \implies \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$\implies \lim_{h \to 0} \dfrac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \implies \lim_{h \to 0} \dfrac{1}{(\sqrt{x+h} + \sqrt{x})} = \dfrac{1}{2\sqrt{x}}$

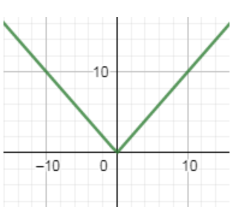
c) $f'(x) = \lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}, \implies \lim_{h \to 0} \dfrac{\frac{3}{x+h} - \frac{3}{x}}{h} \implies \lim_{h \to 0} \dfrac{\frac{3x - 3(x+h)}{x(x+h)}}{h} \implies \lim_{h \to 0} \dfrac{\frac{-3h}{x(x+h)}}{h}$

$\implies \lim_{h \to 0} \dfrac{-3}{x(x+h)} = -\dfrac{3}{x^2}$

Example 2:

1) Differentiability of $y = |x|$ at $x = 0$:

The graph of $y = |x|$:



By definition, $y = |x| \implies y = x$ for $x > 0$ and $y = -x$ for $x < 0$; therefore,

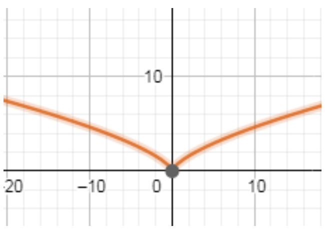
For $x > 0$, $f'(x) = \lim_{h \to 0} \dfrac{(x+h) - x}{h} = 1$;

For $x < 0$, $f'(x) = \lim_{h \to 0} \dfrac{-(x+h) - (-x)}{h} = -1$;

There cannot be derivative at the origin for $y = |x|$ because the one-sided derivatives differ.

2) Differentiability of $y = x^{2/3}$ at $x = 0$

The graph of $y = x^{2/3}$:



$f'(x) = \lim_{h \to 0} \dfrac{(x+h)^{2/3} - x^{2/3}}{h}$ derivative at $x = 0$: $f'(0) = \lim_{h \to 0} \dfrac{(0+h)^{2/3} - 0^{2/3}}{h} \implies f'(0) = \lim_{h \to 0} \dfrac{1}{h^{1/3}}$, but:

$$\lim_{h \to 0^+} \frac{1}{h^{1/3}} = \infty$$

$$\lim_{h \to 0^-} \frac{1}{h^{1/3}} = -\infty$$

There is no derivative at the origin for $y = x^{2/3}$ because the one-sided limits differ.