Limits at Infinity. Asymptotes to graphs

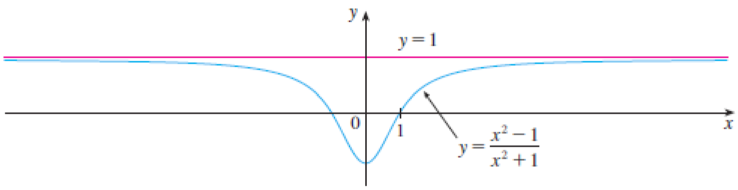
NOTES 05

Instructor: Carlos Sotuyo

Let's begin by investigating the behavior of the function defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large.



As x grows larger and larger you can see that the values of y get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing:

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

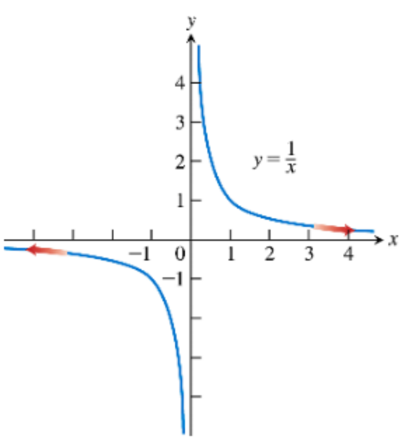
Definition:

Let f be a function defined on some Interval (a, ∞) then,

$$\lim_{x \to \infty} f(x) = L$$

Means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Examine the behavior of the function $f(x) = \dfrac{1}{x}$ as $x \to \infty$ and $x \to -\infty$:



Notice that: $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$

Discuss: Limits at Infinity of Rational Functions

a) $\lim_{x\to\infty} \frac{2x^2+5x+1}{5x^2+3x+10} = \frac{2}{5}$

b) $\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = 0$

Definition:

The line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either,

$$\lim_{x\to\infty} f(x) = b; \quad \text{or,}$$

$$\lim_{x\to-\infty} f(x) = b$$

Discuss the following limits at infinity:

a) $\lim_{x\to-\infty} e^x = 0$

b) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0$

c) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$

d) $\lim_{x\to\infty} x - \sqrt{x^2+16} = 0$

Vertical asymptotes::

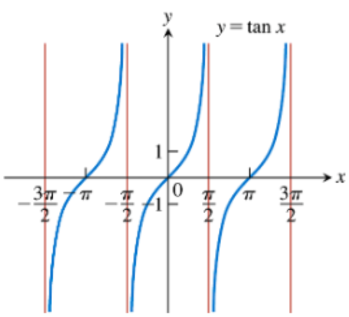
Discuss:

a) $\lim_{x\to 0^+} \frac{1}{x} = \infty$

b) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

c) $\lim_{x\to\pi/2^-} \tan(x) = \infty$

d) $\lim_{x\to\pi/2^+} \tan(x) = -\infty$



Definition:

Vertical asymptotes: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if the function's domain is restricted at $x = a$, and, as a result:

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{OR} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Limits at Infinity of Rational Functions. Solution:

a) $\lim\limits_{x\to\infty} \dfrac{2x^2+5x+1}{5x^2+3x+10} = \dfrac{2}{5}$ Dividing by the largest power of x in the denominator, x^2 :

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{5x}{x^2}+\frac{1}{x^2}}{\frac{5x^2}{x^2}+\frac{3x}{x^2}+\frac{10}{x^2}} \implies \lim_{x\to\infty} \frac{2+\frac{5}{x}+\frac{1}{x^2}}{5+\frac{3}{x}+\frac{10}{x^2}} \implies \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5} \qquad \text{Since,} \quad \lim_{x\to\infty} \frac{k}{x^n} = 0$$

Since, $\lim\limits_{x\to\infty} f(x) = \dfrac{2}{5}$, then $y = \dfrac{2}{5}$ is a horizontal asymptote of $f(x)$. And, because $5x^2+3x+10 \neq 0$ the function has no vertical asymptotes.

b) $\lim\limits_{x\to\infty} \dfrac{11x+2}{2x^3-1} = 0$ Dividing by the largest power of x in the denominator, x^3 :

$$\lim_{x\to\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}} \implies \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} \implies \lim_{x\to\infty} \frac{0+0}{2-0} = 0$$

Since, $\lim\limits_{x\to\infty} f(x) = 0$, then $y = 0$ is a horizontal asymptote of $f(x)$. And, because $2x^3-1=0$ has solution $x = \dfrac{\sqrt[3]{4}}{2}$, and

$$\lim_{x\to\left(\frac{\sqrt[3]{4}}{2}\right)^+} \frac{11x+2}{2x^3-1} = \infty \qquad \text{and} \qquad \lim_{x\to\left(\frac{\sqrt[3]{4}}{2}\right)^-} \frac{11x+2}{2x^3-1} = -\infty$$

then $x = \dfrac{\sqrt[3]{4}}{2}$ is a vertical asymptote.