One-Sided Limits. Continuity.

NOTES 04

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2.4 One-Sided Limits [taken from Thomas Calculus]. In this section we extend the limit concept to one-sided limits, which are limits as x approaches the number c from the left-hand side, where $(x < c)$ or the right-hand side $(x > c)$ only.

$$\lim_{x \to c^+} f(x) = L$$

The symbol $x \to c^+$ means that we consider only values of x greater than c. It is read *x approaches c from the right.*

Similarly, the symbol $x \to c^-$ means that we consider only values of x less than c. It is read *x approaches c from the left.*

$$\lim_{x \to c^-} f(x) = L$$

Notice: we will see that the two sided limits at c not necessarily coincide.

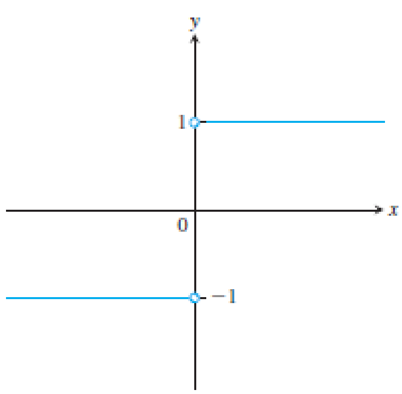
Example 1: Discuss, both analytically and graphically:

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1$$

and

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Consider the graph of the function: $f(x) = \dfrac{|x|}{x}$

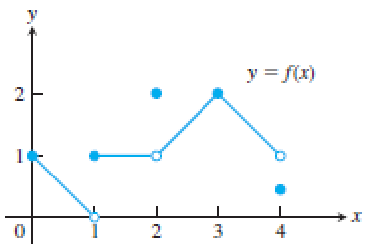


Discuss that

$$\lim_{x \to 0} \frac{|x|}{x} = DNE$$

Does not exist (DNE) since the left side limit $\neq$ right side limit.

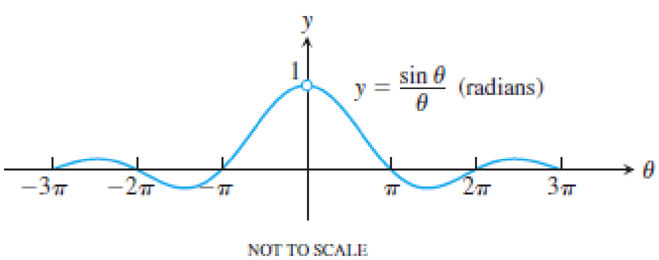
Example 2: Based on the following graph:



Discuss, for $x = c$, where $c = 0, 1, 2, 3, 4$:

a) $f(x)$ b) $\lim_{x \to c^-} f(x)$ c) $\lim_{x \to c^+} f(x)$ d) $\lim_{x \to c} f(x)$

Example 3: Discuss the one sided limits of $f(x) = \dfrac{\sin(\theta)}{\theta}$ as $\theta \to 0$



Theorem:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Example 4. Show that:

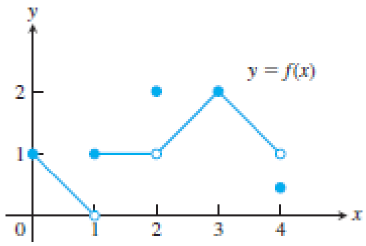
a) $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x} = 0$ b) $\lim_{x \to 0} \dfrac{\tan(x)}{x} = 1$ c) $\lim_{x \to 0} \dfrac{\sin(2x)}{x} = 2$ d) $\lim_{x \to 0} \dfrac{\sin(3x)}{\sin(5x)} = \dfrac{3}{5}$

2.5. Continuity: Intuitively, the graph of a function can be described as a *continuous curve* if it has no breaks or holes.

$$\lim_{x \to c} f(x) = f(c)$$

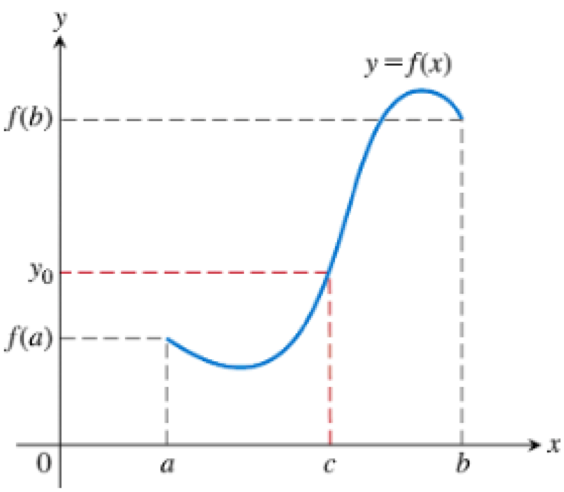
Provided that the limit at c exists, and the function is defined at c.

Discontinuities: holes, jumps, infinite discontinuities. Use graphs to visualize the difference.



Consequence of continuity: Intermediate value Theorem:

If f is continuous on a closed interval $[a, b]$ and y_0 is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number $x = c$ in the interval $[a, b]$ such that $f(c) = y_0$.



Use the IVT to prove the existence of at least one zero or root for a continuous polynomial in a given close interval:

Example: prove that $f(x) = x^3 - x - 1$ has a zero in the close interval $[1, 2]$.

Ans: Knowing that $f(x)$ is continuous everywhere, evaluate the extremes of the interval: $f(1) = -1$ and $f(2) = 5$. Since $y_0 = 0$ lies in between $f(1)$ and $f(2)$, then there exists a number c in $[1, 2]$ such that $f(c) = 0$.