The Precise Definition of a Limit

NOTES 03

Instructor: Carlos Sotuyo

2.3: The Precise Definition of a Limit

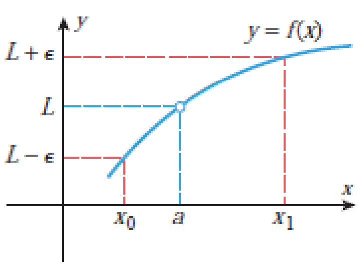
Limit Definition:

Let $f(x)$ be defined for all x in some open interval containing the number a with the possible exception that $f(x)$ need not be defined at a. We will write:

$$\lim_{x\to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that:
$|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

Use a graph to explain the meaning of the previous definition:



This definition, which is attributed to the German mathematician Karl Weierstrass and is commonly called the *epsilon-delta* definition of a two-sided limit, makes the transition from an informal concept of a limit to a precise definition. Specifically, the informal phrase *as close as we like to L* is given quantitative meaning by our ability to choose the positive number ϵ arbitrarily, and the phrase *sufficiently close to a* is quantified by the positive number δ. [Taken from Anton's Calculus].

Example 1: Use the definition to prove that $\lim_{x\to 2}(3x - 5) = 1$

We must show that given any positive number ϵ we can find a positive number δ such that $|(3x-5)-1| < \epsilon$ if $|0 < |x-2| < \delta$. There are two things to do. First, we must discover a value of δ for which this statement holds, and then we must prove that the statement holds for that δ. For the discovery part we begin by simplifying and writing it as:

$|3x - 6| < \epsilon$ if $0 < |x - 2| < \delta$ (1)

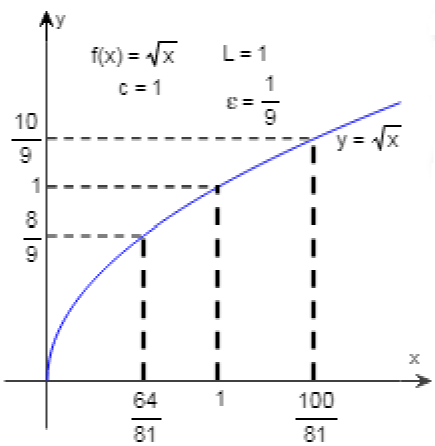
Next we will rewrite this statement in a form that will facilitate the discovery of an appropriate δ:

$3|x-2| < \epsilon$ if $0 < |x-2| < \delta$ (2)

$|x-2| < \epsilon/3$ if $0 < |x-2| < \delta$

It should be self-evident that this last statement holds if $\delta = \epsilon/3$, which completes the discovery portion of our work. Now we need to prove that the initial statement (1) holds for this choice of δ. Since statement (1) is equivalent to (2) and (2) holds with $\delta = \epsilon/3$ so (1) also holds with $\delta = \epsilon/3$. This proves that $\lim_{x\to 2}(3x-5) = 1$.

Example 2: Use the graph below to find the largest value of δ such that for all x, $|f(x)-L| < \epsilon$ whenever $0 < |x-c| < \delta$:



Answer: $\delta = min\left\{1-\frac{64}{81}, \frac{100}{81}-1\right\}$; which yields, $\delta = min\left\{\frac{17}{81}, \frac{19}{81}\right\}$; therefore, $\delta = \frac{17}{81}$.

Note: How do we find the interval $\left(\frac{64}{81}, \frac{100}{81}\right)$?

Work on the absolute value inequality $|f(x)-L| < \epsilon$. In this example, it leads to:

$$|\sqrt{x}-1| < \frac{1}{9}$$

$$-\frac{1}{9} < \sqrt{x}-1 < \frac{1}{9}$$

$$1-\frac{1}{9} < \sqrt{x} < \frac{1}{9}+1$$

$$\frac{8}{9} < \sqrt{x} < \frac{10}{9}$$

$$\left(\frac{8}{9}\right)^2 < x < \left(\frac{10}{9}\right)^2$$

$$\frac{64}{81} < x < \frac{100}{81}$$

Done.