Limits of Functions and Limit Laws

NOTES 02

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Example 1: [Taken form Anton & Biven Calculus, 10th ed.]

Use numerical evidence to show that the value of the following limit is 2:

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

x	0.99	0.999	0.9999	0.99999	–	1.0001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995	–	2.000005	2.000050	2.000500	2.004988

Definition:

We write:

$$\lim_{x\to a} f(x) = L$$

which is read *the limit of f(x) as x approaches a is L.*

Notice that as x approaches 1, the function value approaches 2.

We have used numerical and a graphical approach to analyze limits. Limits may also be found analytically.

Analytically, we can verify that the value of the limit is 2:

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

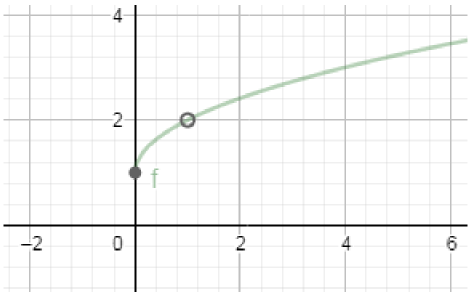
The function $f(x) = \dfrac{x-1}{\sqrt{x}-1}$ is not defined at $x = 1$ When calculating limits we are interested in the function's behavior around c as $x \to c$ (in this case $c = 1$.) Since the function is not defined at 1, let's explore an algebraically equivalent way of writing the given function:

$\dfrac{x-1}{\sqrt{x}-1} \cdot \dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \dfrac{(x-1)(\sqrt{x}+1)}{(x-1)} = \sqrt{x}+1$ so we may say that:

$f(x) = \dfrac{x-1}{\sqrt{x}-1} = \sqrt{x}+1$ for $x \neq 1$. Therefore,

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \sqrt{x}+1 = 2$$

Analyze the graph of $f(x) = \dfrac{x-1}{\sqrt{x}-1}$:



1.2 Computing limits:

Constant function: $\lim_{x \to a} k = k$, For a polynomial: $\lim_{x \to a} p(x) = p(a)$

Explain Theorem 1, Limits Laws:

If L, M, c and k are real numbers, and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then:

(a) The limit of a sum is the sum of the limits: $\lim_{x \to c}(f(x) + g(x)) = L + M$

(b) The limit of a difference is the difference of the limits: $\lim_{x \to c}(f(x) - g(x)) = L - M$

(c) The limit of a product is the product of the limits: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

(d) The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero:
$\lim_{x \to c} \dfrac{f(x)}{g(x)}) = \dfrac{L}{M}, \quad M \neq 0$

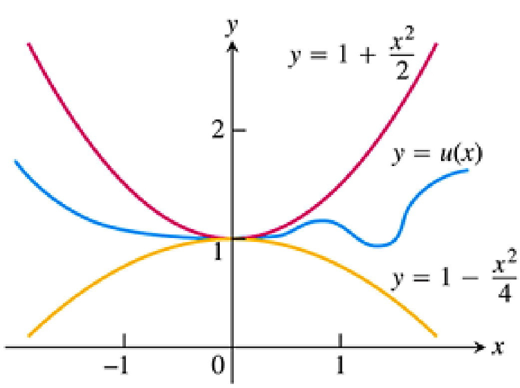
(e) The limit of an nth root is the nth root of the limit: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

(f) The limit of a power is the power of the limit: $\lim_{x \to c} \big(f(x)\big)^n = L^n$

The Sandwich Theorem:

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possible at c itself; suppose also that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$; then, $\lim_{x \to c} f(x) = L$.

Example 4: Any function $u(x)$ whose graph lies in the region between $y = 1 + \dfrac{x^2}{2}$ and $y = 1 - \dfrac{x^2}{4}$ has limit 1 as $x \to 0$.



Since $1 + \dfrac{x^2}{2} \leq u(x) \leq 1 - \dfrac{x^2}{4}$ and $\lim_{x \to 0} 1 + \dfrac{x^2}{2} = \lim_{x \to 0} 1 - \dfrac{x^2}{4} = 1$; then, $\lim_{x \to 0} u(x) = 1$