Continuous Income Stream

Suppose your loving aunt established a trust that pays you \$2,000 a year for 10 years.

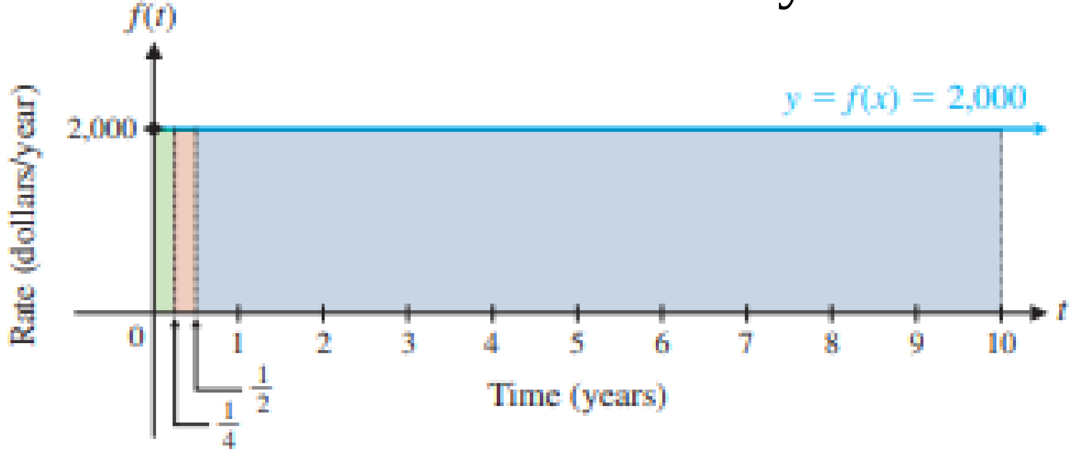
What is the total amount you will receive from the trust by the end of the 10th year?

Since there are 10 payments of \$2,000 each, you will receive $10 \times \$2{,}000 = \$20{,}000$.

Continuous Income Stream

Considering this example from a different perspective, assume that the income stream is continuous at a rate of $2,000 per year.

In the figure, the area under the graph of $f(t) = 2{,}000$ from 0 to t represents the income accumulated t years after the start.



The total income over the 10 year period is given by the definite integral

$$\int_0^{10} 2{,}000\,dt = 2{,}000t\,\Big|_0^{10} = 2{,}000(10) - 2{,}000(0) = \$20{,}000$$

Example 3 Continuous Income Stream

The rate of change of the income produced by a vending machine is given by $f(t) = 5{,}000e^{0.04t}$ where t is the time in years since the installation of the machine.

Find the total income produced by the machine during the first 5 years of operation.

Solution The area under the graph of the rate-of-change function from 0 to 5 gives the total income over the first 5 years. This is found by evaluating the definite integral:

$$\text{Total income } = \int_0^5 5{,}000e^{0.04t}\,dt \quad = 125{,}000e^{0.04t}\Big|_0^5$$

$$= 125{,}000e^{0.04(5)} - 125{,}000e^{0.040}$$

$$= 152{,}675 - 125{,}000 \quad = \$27{,}675$$

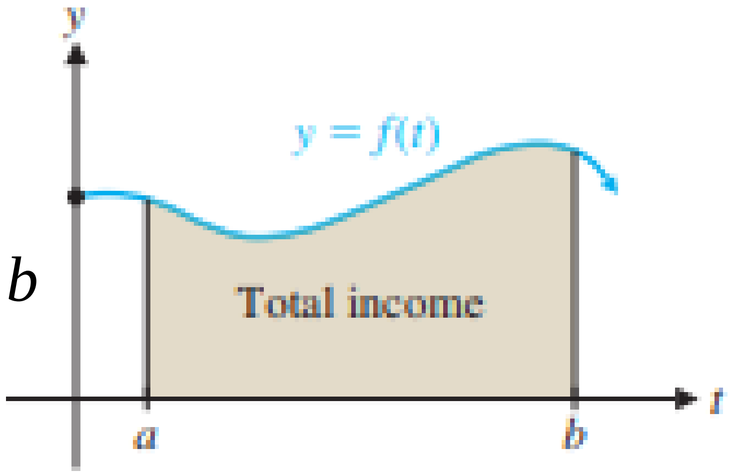
The vending produces total income of $27,675 in the first 5 years.

Definition Total Income for a Continuous Income Stream

If $f(t)$ is the rate of flow of a continuous income stream, then the **total income** produced during the period from $t = a$ to $t = b$ is

$$\text{Total income} = \int_a^b f(t)\,dt$$

In the figure the function $y = f(t)$ represents the rate of flow for a continuous income stream.
The total income between $t = a$ and $t = b$ is the shaded area shown under the curve of $f(t)$.



Future Value of a Continuous Income Stream

The continuous compound interest formula $A = Pe^{rt}$ gives the future amount A, of principal P, compounded continuously at an annual compounding interest rate r (as a decimal), for t years.

If money earns 8% compounded continuously, the future value of a $2,000 investment in 9 years is

$$A = 2,000e^{0.08(9)} = \$4,108.87$$

In the previous example of the trust paying $2,000 per year, the total value of the trust after 10 years, $20,000, is equal to the area under the graph of $f(t) = 2,000$ from 0 to 10.

Future Value of a Continuous Income Stream continued

If at the end of each year, you invest the $2,000 you earned that year at 8% compounded continuously, the amount at the end of 10 years, to the nearest dollar, would be

$$A = 2,000e^{0.08(9)} + 2,000e^{0.08(8)} + 2,000e^{0.08(7)} + \cdots 2,000e^{0.08(0)}$$

$$= 4,108.87 + 3,792.96 + \$3,501.35 + \cdots + 2,000$$

$$= \$29,429$$

This amount underestimates the future value of the *continuous* income stream because deposits are made only once each year, not continuously.

Future Value of a Continuous Income Stream continued

The future value of the continuous income stream is equal to the area under the graph of $2{,}000e^{0.08(10-t)}$ from 0 to 10:

$$FV = \int_0^{10} 2{,}000e^{0.08(10-t)}\,dt = 2{,}000\left.\frac{e^{0.08(10-t)}}{-0.08}\right|_0^{10}$$

$$= 2{,}000\left(\frac{1}{-0.08} + \frac{e^{0.08(10)}}{0.08}\right)$$

$$= \$30{,}639$$

At the end of 10 years, you will have received, \$30,639, including interest.

How much is interest?

Future Value of a Continuous Income Stream continued

Because you received $20,000 in income from the trust, the interest is the difference between the future value $30,639 and the income, $20,000.

$$\$30{,}639 - \$20{,}000 = \$10{,}639$$

is the interest earned by the income received from the trust over the 10-year period.

Definition Future Value of a Continuous Income Stream

If $f(t)$ is the rate of flow of a continuous income stream, $0 \leq t \leq T$, and if the income is continuously invested at a rate r, compounded continuously, then the **future value** ***FV*** at the end of T years is given by

$$FV = \int_0^T f(t)e^{r(T-t)}\,dt = e^{rT}\int_0^T f(t)e^{-rt}\,dt$$

The future value of a continuous income stream is the total value of all money produced by the continuous income stream (income and interest) at the end of T years.

Example 4 Future Value of a Continuous Income Stream

The continuous income rate of flow for the vending machine in Example 3 was $f(t) = 5{,}000e^{0.04t}$.

Find the future value of this income stream at 12%, compounded continuously for 5 years, and find the total interest earned. Round answer to the nearest dollar.

Solution Using the formula

$$FV = e^{rT}\int_0^T f(t)e^{-rt}\,dt$$

with $r = 0.12$, $T = 5$, and $f(t) = 5{,}000e^{0.04t}$

$$FV = e^{0.12(5)}\int_0^5 5{,}000e^{0.04t}e^{-0.12t}\,dt$$

Example 4 Future Value of a Continuous Income Stream continued

$$FV = e^{0.12(5)} \int_0^5 5,000 e^{0.04t} e^{-0.12t}\, dt$$

$$= e^{0.6} \int_0^5 5,000 e^{-0.08t}\, dt$$

$$= 5,000 e^{0.6} \left(\frac{e^{-0.08t}}{-0.08} \right) \Bigg|_0^5$$

$$= 5,000 e^{0.6} (-12.5 e^{-0.4} + 12.5)$$

$$= \$37,545$$

The future value of the income stream at 12% compounded continuously at the end of 5 years is $37,545.

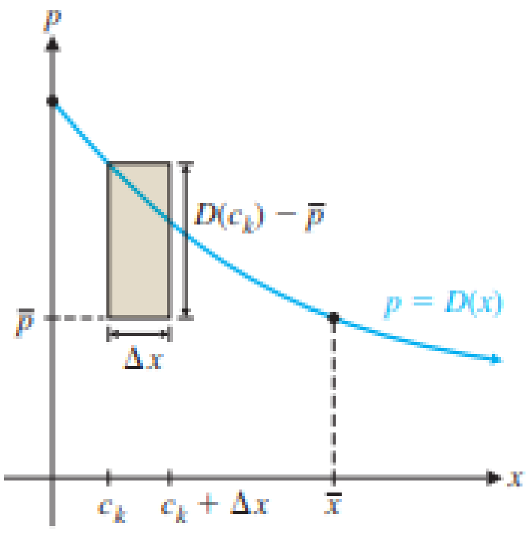
We previously found the total income produced over 5 years was $27,675. The interest earned is $37,545 – $27,675 = $9,870

Consumers' Surplus

Let $p = D(x)$ be the price-demand equation for a product, where x is the number of units of the product that consumers will purchase at a price $\$p$ per unit.

Suppose that $\overline{p}$ is the current price and $\overline{x}$ is the number of units that can be sold at that price.

The price-demand curve in the figure shows that if the price is higher than $\overline{p}$, the demand x is less than $\overline{x}$, but some consumers are still willing to pay the higher price.



Consumers' Surplus continued

Consumers who are willing to pay more than $\overline{p}$, but are still able to buy the product at $\overline{p}$, have saved money. We are to find the total amount saved by all consumers who are willing to pay a price higher than $\overline{p}$ for the product.

Consider the interval $[c_{k,} c_k + \Delta x]$, where $c_k + \Delta x < \overline{x}$. If the price remained constant on the interval, the savings on each unit would be the difference between $D(c_k)$, the price they are willing to pay and $\overline{p}$, the price they actually pay.

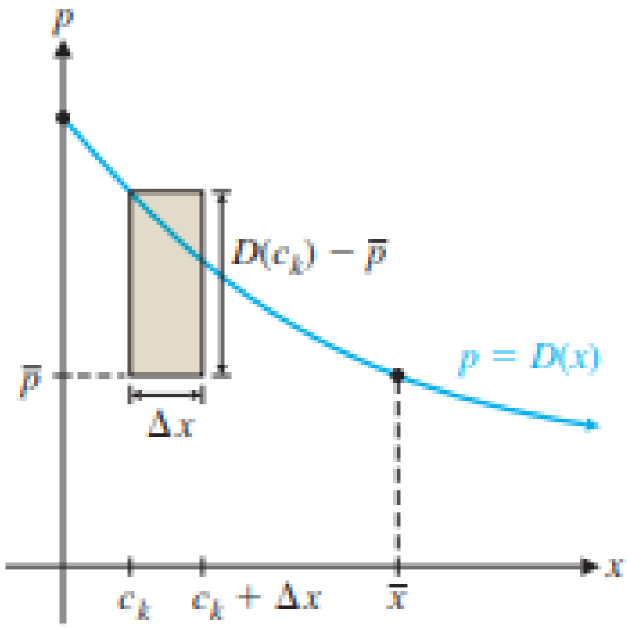
Consumers' Surplus continued

Since Δx represents the number of units purchased over the interval, the total savings on the interval is approximately

$[D(c_k) - \overline{p}]\Delta x$ (savings per unit) × (number of units)

This represents the area of the shaded rectangle in the figure.
If the interval is divided into n equal subintervals, the total savings is found by adding the areas of each corresponding rectangle. This is a Riemann sum for the integral

$$\int_0^{\overline{x}} [D(x) - \overline{p}]\,dx$$



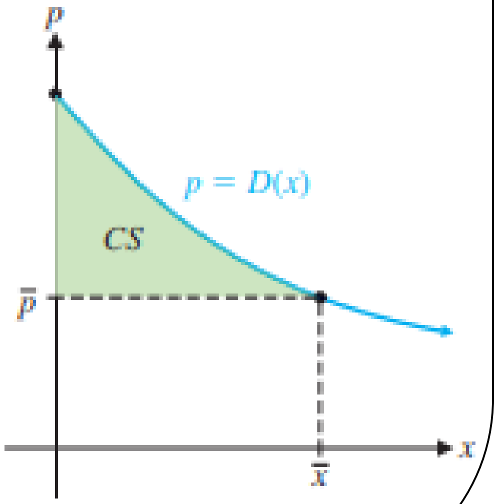
Definition Consumers' Surplus

If $(\overline{x}, \overline{p})$ is a point on the graph of the price-demand equation $p = D(x)$ for a particular product, then the **consumers' surplus** CS at a price level of $\overline{p}$ is

$$CS = \int_0^{\overline{x}} [D(x) - \overline{p}]\,dx$$

which is the area between $p = \overline{p}$ and $p = D(x)$ from $x = 0$ to $x = \overline{x}$, as shown in the figure.

The consumers' surplus represents the total savings to consumers who are willing to pay more than $\overline{p}$ for the product but are still able to buy the product for $\overline{p}$.



Example 5 Consumers' Surplus

Find the consumers' surplus at a price level of $8 for the price-demand equation

$$p = D(x) = 20 - 0.05x$$

Solution Step 1 Find $\bar{x}$, the demand when the price is $\bar{p} = 8$:

$$\bar{p} = 20 - 0.05\bar{x}$$

$$8 = 20 - 0.05\bar{x}$$

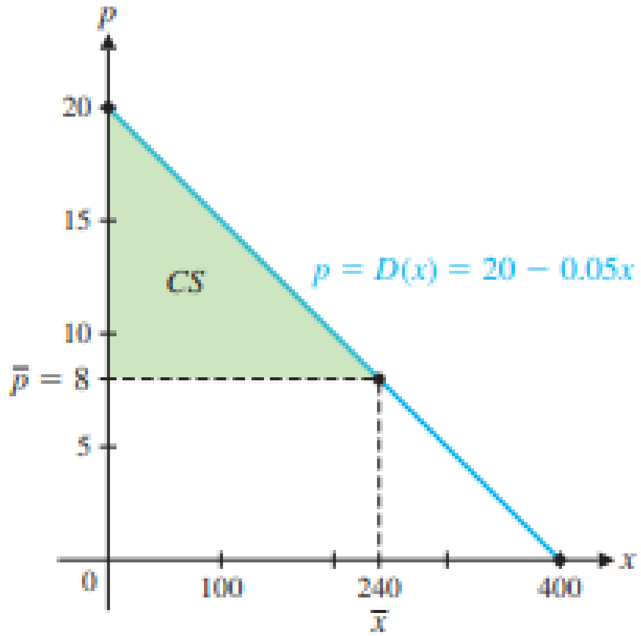
$$0.05\bar{x} = 12$$

$$\bar{x} = 240$$

Example 5 Consumers' Surplus continued

Step 2 Sketch a graph.

This is shown in the figure.



Step 3 Find the consumers' surplus (the shaded area in the graph):

$$CS = \int_0^{\bar{x}} [D(x) - \bar{p}]\,dx$$

$$= \int_0^{240} (20 - 0.05x - 8)\,dx$$

$$= \int_0^{240} (12 - 0.05x)\,dx$$

$$= (12x - 0.025x^2)\Big|_0^{240}$$

$$= 2{,}880 - 1{,}440 = \$1{,}440$$

The total savings to consumers who are willing to pay a higher price for the product is $1,440.

Producers' Surplus

Let $p = S(x)$ be the price-supply equation for a product, $\overline{p}$ is the current price and $\overline{x}$ is the current supply.

Some suppliers are willing to supply some units at a lower price than $\overline{p}$.

The additional money that these suppliers gain from the higher price is called the *producers' surplus* and can be expressed in terms of a definite integral.

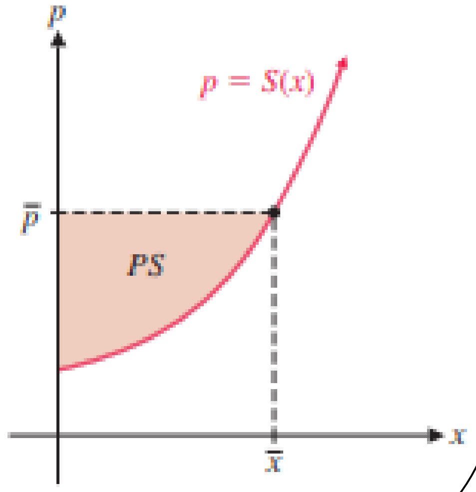
Definition Producers' Surplus

If $(\overline{x}, \overline{p})$ is a point on the graph of the price-supply equation $p = S(x)$ for a particular product, then the **producers' surplus** PS at a price level of $\overline{p}$ is

$$PS = \int_0^{\overline{x}} [\overline{p} - S(x)]\,dx$$

which is the area between $p = \overline{p}$ and $p = S(x)$ from $x = 0$ to $x = \overline{x}$, as shown in the figure.

The producers' surplus represents the total gain to producers who are willing to supply units at a lower price than $\overline{p}$ but are still able to supply units at $\overline{p}$.



Example 6 Producers' Surplus

Find the producers' surplus at a price level of \$20 for the price-demand equation

$$p = S(x) = 2 + 0.0002x^2$$

Solution Step 1 Find $\overline{x}$, the supply when the price is $\overline{p} = 20$:

$$\overline{p} = 2 + 0.0002\overline{x}^2$$

$$20 = 2 + 0.0002\overline{x}^2$$

$$0.0002\overline{x}^2 = 18$$

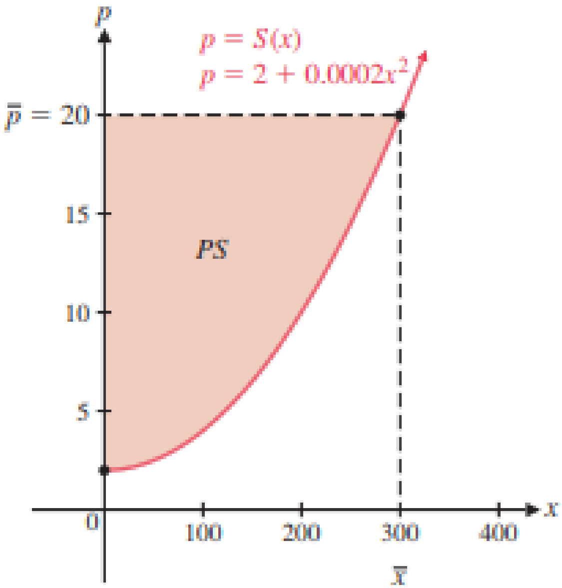
$$\overline{x}^2 = 90{,}000$$

$$\overline{x} = 300$$

Example 5 Consumers' Surplus continued

Step 2 Sketch a graph.

This is shown in the figure.



Step 3 Find the producers' surplus (the shaded area in the graph):

$$PS = \int_0^{\overline{x}} [\overline{p} - S(x)]\,dx$$

$$= \int_0^{300} [20 - (2 + 0.0002x^2)]\,dx$$

$$= \int_0^{300} (18 - 0.0002x^2)\,dx$$

$$= \left(18x - 0.0002\frac{x^3}{3}\right)\Bigg|_0^{300} = 5,400 - 1,800 = \$3,600$$

The total gain to producers who are willing to supply units at a lower price is $3,600.

Equilibrium Price

In a free competitive market, the price of a product is determined by the relationship between supply and demand.

If $p = D(x)$ and $p = S(x)$ are the price-demand and price-supply equations for a product, and if $(\overline{x}, \overline{p})$ is the point of intersection of these two equations, then $\overline{p}$ is called the **equilibrium price** and $\overline{x}$ is called the **equilibrium quantity**.

A stable price at the equilibrium price $\overline{p}$ is the price level that will determine both the consumers' and producers' surplus.

Example 7 Equilibrium Price and Consumers' and Producers' Surplus

Find the equilibrium price and then find the consumers' and producers' surplus at the equilibrium price level, if

$$p = D(x) = 20 - 0.05x \quad \text{and} \quad p = S(x) = 2 + 0.0002x^2$$

Solution Step 1 Find the equilibrium price.

Solve $D(x) = S(x)$

$$20 - 0.05x = 2 + 0.0002x^2$$

$$0.0002x^2 + 0.05x - 18 = 0$$

$$x^2 + 250x - 90{,}000 = 0$$

$$x = 200, -450$$

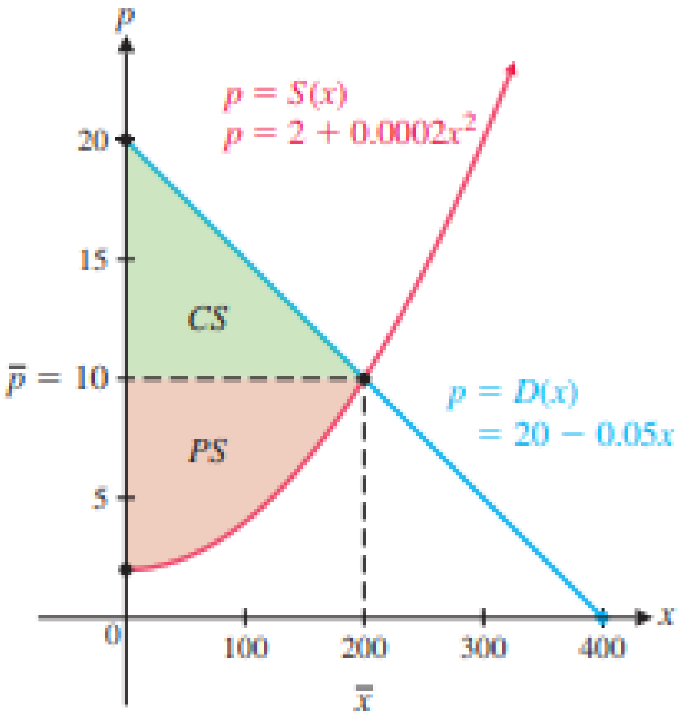
Example 7 Equilibrium Price and Consumers' and Producers' Surplus continued

Since x cannot be negative, the only solution is $\overline{x} = 200$.

With $\overline{x} = 200$, $\overline{p} = D(200) = 10$ and $\overline{p} = S(200) = 10$.

The equilibrium price is $\overline{p} = 10$.

Step 2. The graph is shown in the figure.



Example 7 Equilibrium Price and Consumers' and Producers' Surplus continued

Step 3. Find the consumers' surplus.

$$CS = \int_0^{\bar{x}} [D(x) - \bar{p}]\,dx = \int_0^{200} (20 - 0.05x - 10)\,dx$$

$$= \int_0^{200} (10 - 0.05x)\,dx$$

$$= (10x - 0.025x^2)\Big|_0^{200}$$

$$= 2{,}000 - 1{,}000 = \$1{,}000$$

Example 7 Equilibrium Price and Consumers’ and Producers’ Surplus continued

Step 4. Find the producers’ surplus.

$$PS = \int_0^{\overline{x}} [\overline{p} - S(x)]\,dx = \int_0^{200} [10 - (2 + 0.0002x^2)]\,dx$$

$$= \int_0^{200} (8 - 0.0002x^2)\,dx$$

$$= \left(8x - 0.0002\frac{x^3}{3}\right)\Bigg|_0^{200}$$

$$= 1{,}600 - \tfrac{1{,}600}{3} \approx \$1{,}067$$

Example 7 Equilibrium Price and Consumers' and Producers' Surplus Calculator Process

A graphing calculator can be used to answer this question. Without explanation, the screen shots for this process are: