Notes on Probability:

Fundamental Counting Principle

Definition: If you can choose one item from a group of *M* items and a second item from a group of *N* items, then the total number of two-item choices is $M \cdot N$.

Tree Diagram: A representation of all possible choices.

Ex. 1: You are taking a multiple-choice test that has ten questions. Each of the questions has four answer choices, with one correct answer per question. If you select one of these four choices for each question and leave nothing blank, in how many ways can you answer the questions? Ans: 4^{10}

Ex. 2: Telephone numbers in the United States begin with three- digit area codes followed by seven-digit local telephone numbers. Area codes and local telephone numbers cannot begin with 0 or 1. How many different telephone numbers are possible?

Ans: Area Code Local Telephone Number

8 10 10 8 10 10 10 10 10 10 = 6,400,000,000

Permutations

Permutation is an ordered arrangement of items that occurs when: No item is used more than once. The order of arrangement makes a difference.

Ex3. Permutations of three thing taking all three; taking two at a time.

Factorial Notation: If *n* is a positive integer, the notation *n*! (read "*n* factorial") is the product of all positive integers from n down through 1.

$n! = n(n-1)(n-2)\cdots(3)(2)(1)$

0! (zero factorial), by definition, is 1.

The number of possible permutations of *r* items are taken from *n* items is:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example: You and 19 of your friends have decided to form a business. The group needs to choose three officers– a CEO, an operating manager, and a treasurer. In how many ways can those offices be filled?

Permutations of Duplicate Items

The number of permutations of *n* items, where *p* items are identical, *q* items are identical, *r* items are identical, and so on, is given by:

$$\frac{n!}{p!q!r!\ldots}$$

Example: In how many distinct ways can the letters of the word MISSISSIPPI be arranged?

$$\frac{n!}{p!q!r!} = \frac{11!}{4!4!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!} 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 34,650$$

Combinations: A combination of items occurs when the items are selected from the same group. No item is used more than once. The order of items makes no difference.

Example: ABC, combinations taking two at a time.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example. How many three-person committees could be formed from 8 people?

Example: In December, 2011, the U.S Senate consisted of 51 Democrats and 47 Republicans and 2 Independents. How many distinct five-person committees can be formed if each committee must have 3 Democrats and 2 Republicans?

$${}_{51}C_3 \cdot {}_{47}C_2 = 20,825 \cdot 1081 = 22,511,825$$

Probability*: common sense reduced to calculations (Laplace).*

Probabilities are assigned values from 0 to 1. The closer the probability of a given event is to 1, the more likely it is that the event will occur. The closer the probability of a given event is to 0, the less likely that the event will occur.

Theoretical Probability:

Experiment is any occurrence for which the outcome is uncertain.

Sample space is the set of all possible outcomes of an experiment, denoted by *S*.

Event, denoted by *E* is any subset of a sample space.

Sum of the theoretical probabilities of all possible outcomes is 1.

Computing Theoretical Probability:

If an event *E* has *n*(*E*) equally likely outcomes and its sample space *S* has *n*(*S*) equally-likely outcomes, the theoretical probability of event *E*, denoted by *P*(*E*), is:

P(*E*) = number of outcomes in event *E* = *n*(*E*)

total number of possible outcomes *n*(*S*)

Example: Probability and Combinations: Winning the Lottery

Powerball is a multi-state lottery played in most U.S. states. It is the first lottery game to randomly draw numbers from two drums. The game is set up so that each player chooses five different numbers from 1 to 59 and one Powerball number from 1 to 35. Twice per week 5 white balls are drawn randomly from a drum with 59 white balls, numbered 1 to 59, and then one red Powerball is drawn randomly from a drum with 35 red balls, numbered 1 to 35. A player wins the jackpot by matching all five numbers drawn from the white balls in any order and matching the number on the red Powerball. With one $2 Powerball ticket, what is the probability of winning the jackpot?

$_{59}C_5 \cdot 35 = 5,006,386 \cdot 35 = 175,223,510$

$\frac{1}{175,223,510} \approx 5.707 \times 10^{-9}$

Example: A club consists of five men and seven women. Three members are selected at random to attend a conference. Find the probability that the selected group consists of 3 men.

Probability of an Event Not Occurring

Complement of *E*: If we know *P*(*E*), the probability of an event *E*, we can determine the probability that the event will not occur, denoted by *P*(not *E*).

The probability that an event *E* will not occur is equal to 1 minus the probability that it will occur.

$$P(\text{not } E) = 1 - P(E)$$

The probability that an event *E* will occur is equal to 1 minus the probability that it will not occur.

$$P(E) = 1 - P(\text{not } E)$$

Using set notation, if E' is the complement of *E*, then

$$P(E') = 1 - P(E) \text{ and } P(E) = 1 - P(E')$$

Example: If you are dealt one card from a standard 52-card deck, find the probability that you are not dealt a queen.

Mutually Exclusive Events: Events *A* and *B* are mutually exclusive if it is impossible for them to occur simultaneously.

***Or* Probabilities with Mutually Exclusive Events:**

If *A* and *B* are mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

Example: If one card is randomly selected from a deck of cards, what is the probability of selecting a king or a queen?

Probabilities with Events That Are Not Mutually Exclusive

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ Using set notation, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability to Odds

If we know the probability of an event, we can determine the odds *in favor*, or the odds *against*, the event.

The **odds in favor of *E*** are found by taking the probability that *E* will occur and dividing by the probability that *E* will not occur.

$$\text{Odd in favor of } E = \frac{P(E)}{P(notE)}$$

The **odds against E** are found by taking the probability that E will not occur and dividing by the probability that E will occur.

$$\text{Odd against } E = \frac{P(\text{not } E)}{P(E)}$$

The odds against *E* can also be found by reversing the ratio representing the odds in favor of *E*

Example: You roll a single, six-sided die. Find the odds in favor of rolling a 2

Odds to Probability

If the odds in favor of event *E* are *a* to *b*, then the probability of the event is given by:

$$P(E) = \frac{a}{a+b}.$$

Example: The odds in favor of a particular horse winning a race are 2 to 5. What is the probability this horse will win the race?

Events Involving *And;* Conditional Probability

Independent Events: Two events are independent events if the occurrence of either of them has no effect on the probability of the other.

***And* Probabilities with Independent Events**

If *A* and *B* are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example: A U.S. roulette wheel has 38 numbered slots (1 through 36, 0, and 00). 18 are black, 18 are red, and 2 are green. The ball can land on any slot with equal probability. What is the probability of red occurring on 2 consecutive plays?

$$P(\text{red and red}) = P(\text{red}) \cdot P(\text{red}) = \frac{9}{19} \cdot \frac{9}{19} = \frac{81}{361} \approx 0.224$$

Example: If the probability that South Florida will be hit by a hurricane in any single year is 5/19,

a. What is the probability that South Florida will be hit by a hurricane in three consecutive years?

Dependent Events: Two events are dependent events if the occurrence of one of them has an effect on the probability of the other.

Example: Three people are randomly selected, one person at a time, from 5 freshmen, 2 sophomores, and 4 juniors. Find the probability that the first two people selected are freshmen and the third is a junior.

Example: A letter is randomly selected from the letters of the English alphabet. Find the probability of selecting a vowel, given that the outcome is a letter that precedes h.

S = {a, b, c, d, e, f, g}.

Example: Mammography Screening on 100,000 U.S. Women, Ages 40 to 50

	Breast Cancer	No Breast Cancer	Total
Positive Mammogram	720	6,944	7,664
Negative Mammogram	80	92,256	92,336
Total	800	99,200	100,000

Assuming that these numbers are representative of all U.S. women age 40 to 50, find the probability that a woman in this age range has a positive mammogram, given that she does not have breast cancer.

$$P(\text{positive mammogram}|\text{no breast cancer}) = \frac{6944}{99,200} = 0.07$$