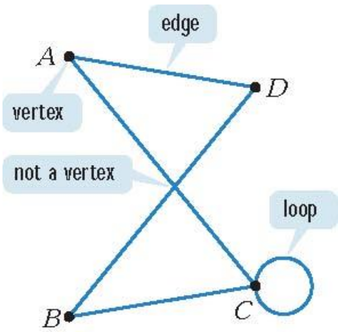
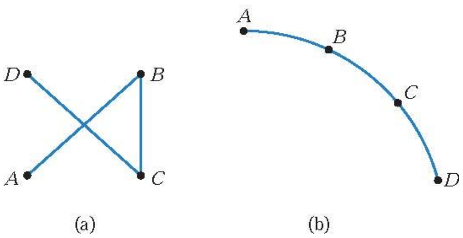
Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices. A loop is a special type of edge that connects a vertex to itself**.**



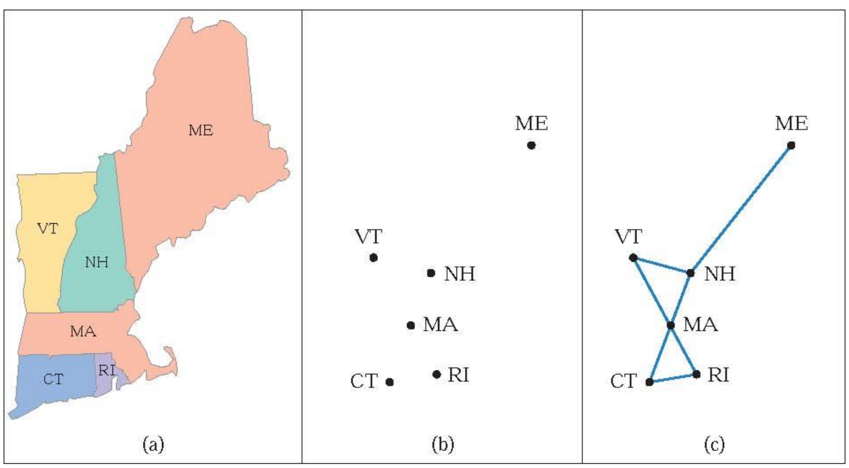
Example: Explain why the figures below show equivalent graphs.



Solution: In both figures, the vertices are *A*, *B*, *C*, and *D*. Both graphs have an edge that connects vertex *A* to vertex *B*, an edge that connects vertex *B* to vertex *C*, and an edge that connects vertex *C* to vertex *D*. Because the two graphs have the same number of vertices connected to each other in the same way, they are equivalent.

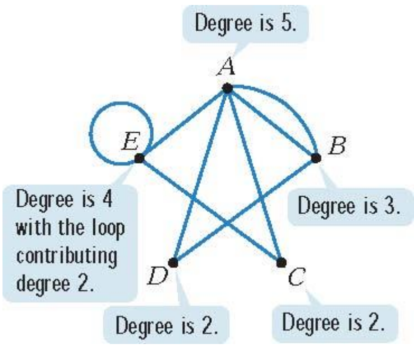
Example: Modeling Bordering Relationships for the New England States

The map of New England states are given below. Draw a graph that models which New England states share a common border. Use vertices to represent the states and edges to represent common borders.



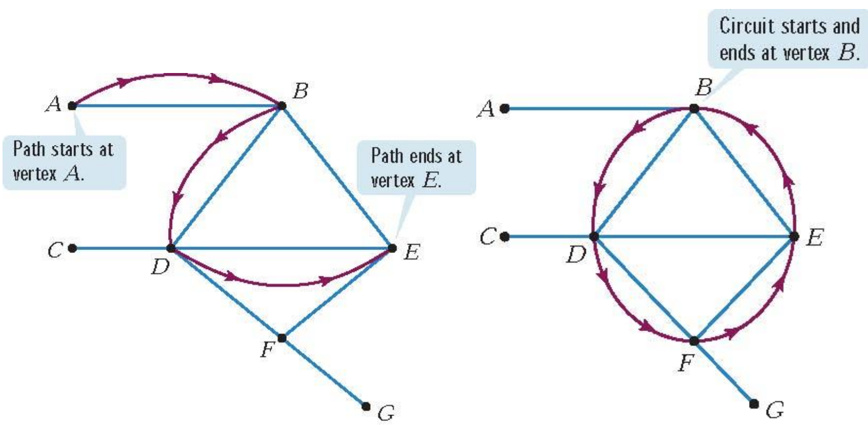
Circuits & Paths Vocabulary of Graph Theory

The degree of a vertex is the number of edges at that vertex. A vertex with an even number of edges attached to it is an even vertex. A vertex with an odd number of edges attached to it is an odd vertex. Two vertices in a graph are said to be adjacent vertices if there is at least one edge connecting them. A loop is incident to only one vertex, when measuring the degree of such a vertex, the loop is counted twice.



Path
A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices.

Circuit
A circuit is a path that begins and ends at the same vertex.



An **Euler path** is a path that uses every edge of a graph exactly once.

An **Euler circuit** is a circuit that uses every edge of a graph exactly once.

Emphasis:

An Euler path starts and ends at different vertices. An Euler circuit starts and ends at the same vertex.

***For every vertex V other than the starting and ending vertices, the path P enters V the same number of times that it leaves V. That is, all vertices other than the two endpoints of P must be even vertices.

Therefore,

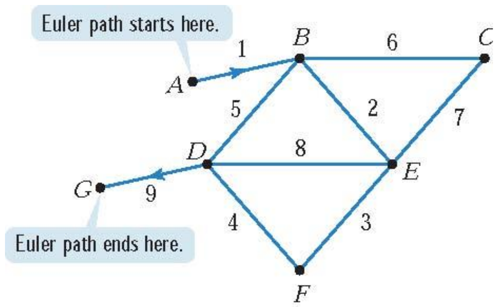
If a graph G has an Euler path, then it must have exactly two odd vertices. Or, to put it another way, if the number of odd vertices in G is anything other than 2, then G cannot have an Euler path.

If a graph G has an Euler circuit, then all of its vertices must be even vertices.

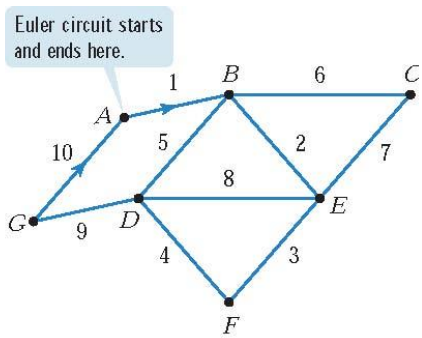
Euler's Path and Circuit Theorems
A graph will contain an Euler path if it contains at most two vertices of odd degree.
A graph will contain an Euler circuit if all vertices have even degree.

Example: Euler path:

Recall: An ***Euler path*** is a path that travels through *every edge* of a graph once and only once. Each edge must be traveled and no edge can be retraced.



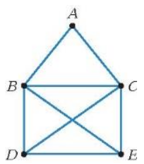
Example 3: Euler circuit: An ***Euler circuit*** is a circuit that travels through every edge of a graph once and only once. Like all circuits, an Euler circuit must begin and end at the same vertex.



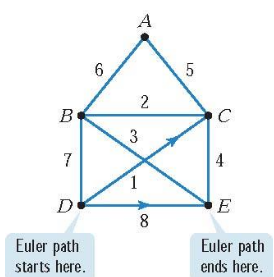
The following statements are true for connected graphs:

1. If a graph has exactly two odd vertices, then it has at least one Euler path, but no Euler circuit. Each Euler path must start at one of the odd vertices and end at the other one.
2. If a graph has no odd vertices (all even vertices), it has at least one Euler circuit (which, by definition, is also an Euler path). An Euler circuit can start and end at any vertex.
3. If a graph has more than two odd vertices, then it has no Euler paths and no Euler circuits.

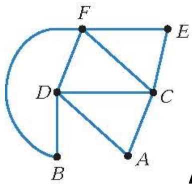
Example 4: Explain why the graph in the figure has at least one Euler path. Use trial and error to find one such path.



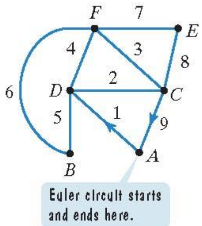
Solution: Count the number of edges at each vertex to determine if the vertex is odd or even. We see from the figure on the left that there are exactly two odd vertices, *D* and *E*. Thus, by the first statement in Euler's theorem, the graph has at least one Euler path, but no Euler circuit. Euler's Theorem says that a possible Euler path must start at one of the odd vertices and end at the other one. To do this, we use trial and error.



Example 5: The figure has at least one Euler circuit. Find one by Fleury's Algorithm. (Notice, the figure has at least one Euler circuit because the graph has *no* odd vertices.)



Answer:



Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one? While it usually is possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm:**

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges. When faced with a choice of edges to trace, first choose any edge that is not a bridge. [A *bridge* is an edge whose removal would make the graph disconnected.] Travel over a bridge only if there is no alternative.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

Example: The figure has at least one Euler circuit. Find one by Fleury's Algorithm.