Selected miscellaneous exercises 11

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Suggested solutions for Miscellaneous exercises 11, problems 5, 16, 17, 19 and 20 page 172 from Pure Mathematics 1, by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

5.

Note: in the process of composition the output of one function becomes the input of another. Repeated composition of such a function with itself is called iterated function. A function iterated n times is sometimes denoted $f^n(x)$. And,trivial but so important, the composition of any given function, let's say, $g(x)$ with the function $f(x) = x$ yields $g(x)$.

(a) $f(x) = x^{\frac{1}{3}} + 10$ therefore composition of f with f is given by: $ff : x \mapsto (x^{\frac{1}{3}} + 10)^{\frac{1}{3}} + 10$ then ff(-8)= $ff : (-8) \mapsto ((-2^3)^{\frac{1}{3}} + 10)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (-2 + 10)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (8)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (2^3)^{\frac{1}{3}} + 10$ $= 12$;

(b) Inverse function of f:

$$y = x^{\frac{1}{3}} + 10$$

$$(y - 10)^3 = x$$

$$f^{-1} : x \mapsto (x - 10)^3$$

$$f^{-1} : 13 \mapsto (13 - 10)^3 = 27.$$

16.

$f : \mapsto 2x + 7$ and $g : \mapsto x^3 - 1$; then,

(a) f^{-1}

$$y = 2x + 7$$

$$y - 7 = 2x$$

$$\frac{y - 7}{2} = x,$$

Therefore, we rename this function as the inverse of the original:

$f^{-1} = \dfrac{x - 7}{2} = \dfrac{1}{2}(x - 7).$

(b) g^{-1}

$$y = x^3 - 1$$

$$y + 1 = x^3$$

$$\sqrt[3]{y + 1} = x$$

The inverse is of g is $g^{-1} = \sqrt[3]{x+1}$

c) The composition of f^{-1} with g^{-1}:

$$g^{-1}f^{-1} = \sqrt[3]{\frac{x-7}{2}+1}$$

$$g^{-1}f^{-1} = \sqrt[3]{\frac{x}{2}-\frac{7}{2}+1}$$

$$g^{-1}f^{-1} = \sqrt[3]{\frac{x}{2}-\frac{5}{2}}$$

$$g^{-1}f^{-1} = \sqrt[3]{\frac{1}{2}(x-5)}$$

(d) $f^{-1}g^{-1}$ This is the composition of g^{-1} with f^{-1} :

$$f^{-1}g^{-1} = \frac{\sqrt[3]{x+1}-7}{2} = \frac{1}{2}(\sqrt[3]{x+1}-7)$$

(e)$fg = 2(x^3-1)+7 = 2x^3-2+7 = 2x^3+5.$

(f)$gf = (2x+7)^3-1.$

(g) $(fg)^{-1}$ denotes the inverse of the composition of fg : —see item (g):

$$y = 2x^3+5$$

Then, solving for x, we have:

$$\frac{1}{2}(y-5) = x^3$$

$$\sqrt[3]{\frac{1}{2}(y-5)} = x$$

Therefore the inverse of (fg) is given by

$$(fg)^{-1} = \sqrt[3]{\frac{1}{2}(x-5)}$$

(h)$(gf)^{-1}$ denotes the inverse of the composition of gf :

Per item (f), $(gf) = (2x+7)^3-1$, then, solving or x:

$$y = (2x+7)^3-1$$

$$y+1 = (2x+7)^3$$

$$\sqrt[3]{y+1} = 2x+7$$

$$\frac{1}{2}\left(\sqrt[3]{y+1}-7\right) = x$$

$$(gf)^{-1} = \frac{1}{2}\left(\sqrt[3]{x+1}-7\right)$$

17.

$f :\mapsto 10-x$, for all real numbers. Let's consider the composition of f with itself (iteration):

$$ff :\mapsto 10-(10-x) = x$$

$$fff \;\; \texttt{or} \;\; f^3 = 10 - x$$
$$f^4 : \mapsto 10 - (10 - x) = x$$
$$f^5 = 10 - x$$

That is, the composition of f —or iteration of f — an odd number of times is given by: $f^{odd} = 10 - x$ while $f^{even} = x$; then:

(a)$f(7) = 10 - 7 = 3$
(b)$f^2(7) = 7$
(c)$f^{15}(7) = 10 - 7 = 3$
(d)$f^{100}(7) = 7$

19.

Note: By definition a function f that takes x to y, (f). An inverse function is another function that takes y, (f) back to x. This is, a function composed with its inverse function yields the original starting value: $f(f^{-1}) = x$ and $f^{-1}(f) = x$.

(a) given $f(x) = \dfrac{2x-4}{x}$ the composition of the function with itself, ff,denoted as f^2 is given by: $f^2 = \dfrac{2\left[\dfrac{2x-4}{x}\right] - 4}{\left[\dfrac{2x-4}{x}\right]}$

$= \dfrac{4x - 8 - 4x}{x} \cdot \dfrac{x}{2x-4} = \dfrac{-8}{2x-4} = \dfrac{-4}{x-2} = \dfrac{4}{2-x}$.

(b) $y = \dfrac{2x-4}{x}$ the inverse is given by: $yx = 2x - 4$, or:

$$4 = 2x - yx$$
$$4 = x(2-y)$$
$$\frac{4}{2-y} = x$$

we have obtained the inverse function: a formula that transform what we may called the old y into the new x

Let's rename this function as the inverse of the original $f(x)$, as follows:

$$f^{-1} = \frac{4}{2-x}$$

(c) Answers to (a) and (b) prove that, for the given function, $ff = f^{-1}$; then fff which is the composition of f with ff is equivalent to the composition of f with its inverse f^{-1} which yield x (see note above). Therefore:

$$f^3 = fff = f^{-1}f = x$$

(d) Since $fff = x$ then f^4 is the composition of $f(x) = \dfrac{2x-4}{x}$ with x, which yields $f(x)$ itself.

(e) Whenever the composition of f with itself is a multiple of 3, the iteration (repeated composition of the function with itself) yields x. Let's approach the pattern of the composition of f this way: $f^{12} = fff\ fff$
fff, that is, every three compositions of f with itself it yields x, and, of course, the composition of x with the x is x.

(f) f^{82}; since $82 = 3(27) + 1$ it means that, while $f^{81} = x$ (see previous result) f^{82} is given by the composition of $f(x)$ with x which, of course, yields $f(x) = \dfrac{2x-4}{x}$.

20.

(a) A function is self-inverse if $ff = x$; for $f : \mapsto \dfrac{x+a}{x-1}$ the composition ff yields:

$$ff = \frac{\dfrac{x+a}{x-1} + a}{\dfrac{x+a}{x-1} - 1}$$

$$ff = \frac{\dfrac{x+a+a(x-1)}{x-1}}{\dfrac{x+a-(x-1)}{x-1}}$$

$$ff = \frac{a+a+ax-a}{x-1} \cdot \frac{x-1}{a+a-x+1}$$

$$ff = \frac{x+ax}{x-1} \cdot \frac{x-1}{a+1}$$

$$ff = \frac{x(1+a)}{(1+a)} = x.$$

Therefore the function $f : \mapsto \dfrac{x+a}{x-1}$ is a self-inverse, since $f(f^{-1}) = x$ and the composition of the function with itself also yields x.