Suggested solutions for selected exercises chapter 3 of Pure Mathematics 1 by Hugh Neil and Douglas Qaling.

1.

The domain of the function is the set of numbers for which the function is defined.

a) \( \frac{1}{1 + \sqrt{x}} \) Since the square root of a negative number is not defined in the set of real numbers, the domain of this function is all \( x \geq 0 \).

b) \( \frac{1}{(x - 1)(x - 2)} \); the divisibility by zero has no meaning; or, we say, it is undefined. The answer can be stated as follow: Domain of the function is all \( x \), such as \( x \neq 1 \) and \( x \neq 2 \).

2.

The Domain of these functions are the set of all positive real numbers. That is, negative numbers are not considered in the input (as admissible values for \( x \)). The range of the given function are:

a) for \( f(x) = -5x \) Since the input we consider are positive values of \( x \), the output will always be negative. Answer: \( y < 0 \).

b) for \( f(x) = 3x - 1 \); since the input we consider are positive values of \( x \), being zero the smallest, the output will be \( y > -1 \).

c) for \( f(x) = (x - 1)^2 + 2, y \geq 2 \).

3.

Note: when we take the reciprocal of both sides of an inequality, the sign of the inequality is reversed provided \( a, b, c, d > 0 \). And this is why:

\[
\frac{a}{b} > \frac{c}{d}
\]

Multiply both sides of the inequality by \( bd \):

\[
ad > bc
\]

Divide both sides of the inequality by \( ac \):

\[
\frac{d}{c} > \frac{b}{a}
\]

This result is equivalent to:

\[
\frac{b}{a} < \frac{d}{c}
\]

which is the reciprocal of the original inequality and the inequality sign has been inverted.

a) \( x^{-4} \geq 100 \) it is the same as: \( \frac{1}{x^4} \geq 10^2 \)
Taking the reciprocal of the inequality:

\[ x^4 \leq \frac{1}{10^2} \]

We need to know that, the nth (even) root of a number is the modulus of the number: \( \sqrt[n]{x} = |x| \), again, provided \( n \) is even. So, in our case we get:

\[ |x| \leq \frac{\sqrt{10}}{10} \]

which leads to:

\[ -\frac{\sqrt{10}}{10} \leq x \leq \frac{\sqrt{10}}{10}. \]

b) \( 8x^{-4} < 0.00005 \).

\[
\begin{align*}
8x^{-4} &< 5 \cdot 10^{-5} \\
\frac{8}{x^4} &< \frac{5}{10^5} \\
\frac{x^4}{8} &> \frac{10^5}{5} \\
x^4 &> \frac{8 \cdot 10^5}{5} \\
x^4 &> 2^4 \cdot 10^4 \\
|x| &> 20
\end{align*}
\]

Therefore \( x > 20 \) or \( -x > 20 \) which is equivalent to: \( x > 20 \) or \( x < -20 \).

4.

In order to sketch the graphs of the functions, draw a \( xy \) coordinate system using \( k \) as a unit. This is, in the positive side of both axes (\( x \) and \( y \)) we have \( k, 2k, 3k, \) etc; on the negatives, \( -k, -2k, -3k \) etc. Also, let’s pay attention to the highest power of \( x \) and the sign (positive or negative) of the highest coefficient of \( x \).

Then, in a) the parabola opens upwards, the zeros are \(-4k \) and \(-2k \).

b) The power of \( x \) is 3, so the curve takes the shape of the parent function \( y = x^3 \) with zeros at 0, \( k \) and \( 5k \).

5.

We are given the equation of three curves (parabolas). Let’s label them as \( y_1, y_2 \) and \( y_3 \). All three curves have one point in common. We set pair of equations in order to determine the point in common between two of them, as follows: \( y_1 \) and \( y_2 \); then \( y_1 \) and \( y_3 \) and finally, \( y_2 \) and \( y_3 \), which are all possible combinations.

Equating \( y_1 \) and \( y_2 \) we get:

\[ 2x^2 + 5x = x^2 + 4x + 12 \quad \text{or} \quad x^2 + x - 12 = 0 \quad \text{which factorizes as} \quad (x + 4)(x - 3) = 0 \]

Therefore the two curves have two \( x \) coordinate in common: \( x = -4 \) and \( x = 3 \).

let’s pair \( y_1 \) and \( y_3 \):

\[ 2x^2 + 5x = 3x^2 + 4x - 6 \]
\[ x^2 - x - 6 = 0 \]
Whose factors are: 

\[(x - 3)(x + 2) = 0\]

given two x-values in common between these two curves: 

\[x = 3\] and \[x = -2\]. Indeed we have found that \(x = 3\) is a common x-coordinate; still, let’s verify that \(y_2\) and \(y_3\) have the same point in common:

\[
x^2 + 4x + 12 = 3x^2 + 4x - 6
\]

\[
2x^2 - 18 = 0
\]

Or \(x = \pm 3\). Again, we have found that \(x = 3\) is a common value for all three curves. The y-coordinate can be calculated by substituting in either of the original equations; for instance

\[f(3) = 2(3)^2 + 5(3) = 33\].

The common point is \((3, 33)\).

6.

Since both curves meet at \((-2, 12)\) by evaluating \(f(-2) = 12\) in the equations we can determine the values of \(c\) and \(k\):

\[
f(-2) = 12 = (-2)^2 - 3(-2) + c \quad or \quad c = 2
\]

\[
f(-2) = 12 = k - (-2) - (-2)^2 \quad or \quad k = 14
\]

Therefore our functions are:

\[y = x^2 - 3x + 2 \quad and \quad y = 14 - x - x^2\]

In order to determine the other point at which the two curves meet, we set them equal to one another:

\[
14 - x - x^2 = x^2 - 3x + 2
\]

\[
2x^2 - 2x - 12 = 0
\]

\[(x - 3)(x + 2) = 0 \quad therefore \quad x_1 = -2 \quad and \quad x_2 = 3.
\]

\(x_1 = -2\) was already known to us; so the other point has y-coordinate = 2 since \(f(3) = 2\), the point is \((3, 2)\).

7.

The straight line \(y = x - 1\) meets the curve \(y = x^2 - 5x - 8\) at the points A and B. The curve \(y = p + qx - 2x^2\) also passes through A and B. We are asked to find the values of \(p\) and \(q\).

Setting the two first equations equal to each other, we are able to find A and B:

\[x - 1 = x^2 - 5x - 8 \quad which \ reduces \ to \quad x^2 - 6x - 7 = 0 \quad or \quad (x - 7)(x + 1) = 0
\]

The x-values at which the curves meet are: \(x_1 = 7\) and \(x_2 = -1\). Since \(f(7) = 6\) and \(f(-1) = -2\) points A and B are: \((7, 6)\) and \((-1, -2)\).

Now, by evaluating those points in \(y = p + qx - 2x^2\) we get:

\[f(7) = 6 = p + 7q - 2(7)^2 \quad or \quad p + 7q = 104
\]

\[f(-1) = -2 = p + q(-1) - 2(-1)^2 \quad or \quad p - q = 0
\]

By solving this system of two unknowns, we get: \(8p = 104\) or \(p = 13\), therefore \(q = 13\).

8.

In order to the points at which the line \(10x - 9\) meets the curve \(y = x^2\), again we set the equation equal to each other:

\[10x - 9 = x^2
\]

\[x^2 - 10 + 9 = 0
\]

\[(x - 9)(x - 1) = 0
\]

Therefore \(x_1 = 9\) and \(x_2 = 1\). \(f(9) = 81\) and \(f(1) = 9\). The points of intersection are: \((9, 81)\) and \((1, 9)\).