Selected questions & answers

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Questions taken from Pure Mathematics 1, by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

Miscellaneous exercises 1, page 15, No. 1.

1. Show that the triangle formed by the points (-2,5), (1, 3) and (5, 9) is right-angled.

We have to prove that points (-2, 5), (1, 3) and (5, 9) determine a right triangle:
Method 1: Label (-2, 5) as A, (1, 3) as B and (5, 9) as C. Then the gradient of the line (side of triangle) AB $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)} = -\dfrac{3}{2}$. Gradient of BC, $m = \frac{3}{2}$; since the gradients are the negative reciprocal of one another, side (line) AB$\perp$BC.
Method 2: Converse of Pythagorean Theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.
Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
The distance between (-2, 5) and (1, 3) $= \sqrt{13}$; Distance between (1, 3) and (5, 9) $= \sqrt{52}$ and the distance between (-2, 5)and (5, 9) $= \sqrt{65}$
Since $(\sqrt{65})^2 = (\sqrt{13})^2 + (\sqrt{52})^2$ then, by the converse of the Pythagorean Theorem the triangle is a right triangle.

Miscellaneous exercises 4, page 62, No. 4

4. For what values of k does the equation $2x^2 - kx + 8 = 0$ have a repeated root?

The repeated root occurs when when the discriminant $b^2 - 4ac$ is equal to zero, for any quadratic equation of the form: $ax^2 + bx + c = 0$.

The quadratic equation given is $2x^2 - kx + 8 = 0$ where $b = k$. The discriminant is given by:

$$k^2 - 4(2)(8) = 0$$
$$k^2 - 64 = 0$$
$$(k - 8)(k + 8) = 0$$

Therefore, the values of k for which the quadratics has a repeated roots are:

$$k = 8 \quad \texttt{and} \quad k = -8$$

Exercises 6D, page 87, No. 12
12. A curve has equation $y = x(x - a)(x + a)$, where a is a constant. Find the equations of the tangents to the graph at the points where it crosses the x-axis.

The curve has equation $y = x(x - a)(x + a)$ where a is a constant crosses the x-axis $(y = 0)$ at $x = 0$, $x = a$ and $x = -a$. That is, the points are: $(0, 0)$; $(a, 0)$ and $(-a, 0)$.

The equation of the curve in factored form is equivalent to: $y = x(x^2 - a^2)$ or $y = x^3 - a^2x$. Its derivative is: $y' = 3x^2 - a^2$.

Equations of tangent lines to the curve are given by the gradient $m = (f'(x_0))$ of the tangent at each point (x_0, y_0). The general form of the equation of the tangent line is $y - y_0 = m(x - x_0)$; therefore,

* Equation of tangent line at $(0, 0)$ has gradient $f'(0) = -a^2$: and equation $y - 0 = -a^2(x - 0)$ or $y = -a^2x$.

* Equation of tangent line at $(a, 0)$ has gradient $f'(a) = 2a^2$: and equation $y - 0 = 2a^2(x - a)$ or $y = 2a^2x - 2a^3$.

* Equation of tangent line at $(-a, 0)$ has gradient $f'(-a) = 2a^2$: and equation $y - 0 = 2a^2(x + a)$ or $y = 2a^2x + 2a^3$.

Exercise 6E, page 93, No. 2

2. Find the derivative of the function $f(x) = x^8$ at $x = p$. (Use the difference of two squares formula on $q^8 - p^8$ as often as you can.)

Recall that, the derivative f'(p) is the limit of the slope of secant PQ as Q approaches P. Hence, the derivative of $f(x) = x^8$ at $x = p$ is given by:

$$f'(x) = \lim_{q \to p} \frac{q^8 - p^8}{q - p}$$

$q^8 - p^8$ can be seen as the difference of two squares, $(q^4)^2 - (p^4)^2$ which expands as: $(q^4 - p^4)(q^4 + p^4)$; similarly, $q^4 - p^4$ can be treated as the difference of two squares, $q^4 - p^4 = (q^2 - p^2)(q^2 + p^2)$; then we expand $q^2 - p^2$ to $(q - p)(q + p)$. The complete factored expression is:

$$f'(x) = \lim_{q \to p} \frac{(q - p)(q + p)(q^2 + p^2)(q^4 + p^4)}{q - p}$$

Which simplifies to:

$$f'(x) = \lim_{q \to p} (q + p)(q^2 + p^2)(q^4 + p^4)$$

Then, in the limits, "p" takes the values of "p",

$$f'(p) = 2p \cdot 2p^2 \cdot 2p^4 = 8p^7$$

Miscellaneous exercises 6, page 94, No.11

11. Show that the curves $y = x^3$ and $y = (x + 1)(x^2 + 4)$ have exactly one point in common, and use differentiation to find the gradient of each curve at this point.

We are told that the curves $y = x^3$ and $y = (x + 1)(x^2 + 4)$ have one point in common. Then, it has to be true that, setting equations equal to each other, only one solution can be found; this is:

$$x^3 = (x+1)(x^2+4)$$
$$x^3 = x^3 + 4x + x^2 + 4$$
$$0 = x^3 + x^2 + 4x + 4 - x^3$$
$$0 = x^2 + 4x + 4 = (x+2)^2$$

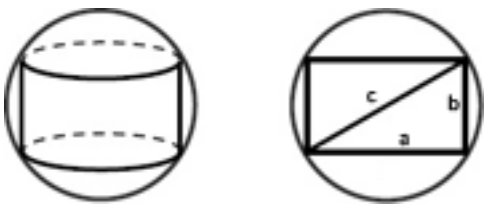
The solution is $x = -2$ and $f(-2) = (-2)^3 = -8$; hence, the common point is $(-2, -8)$.

** The gradient of $y = x^3$ at $(-2, -8)$ is given by $f'(-2) = 3(-2)^2 = 12$.

*** The gradient of $y = (x+1)(x^2+4)$ at $(-2, -8)$ is given by $f'(-2) = 3(-2)^2 + 2(-2) + 4 = 12$.

Excercises 7C, page 110, No.16

16. A circular cylinder is to fit inside a sphere of radius 10 cm . Calculate the maximum possible volume of the cylinder. (It is probably best to take as your independent variable the height, or half the height, of the cylinder.)



The radius of the sphere is equal to 10 cm; therefore, the diagonal $c = 20\ cm$. a is twice the diameter of the cylinder or $= 2r$ (r denotes the radius of the cylinder). b is the height of the cylinder, denoted by h. Hence,by the Pythagorean theorem:

$$20^2 = (2r)^2 + h^2$$
$$400 = 4r^2 + h^2$$

Solving for the radius of the cylinder:

$$r^2 = \frac{400 - h^2}{4}$$

The previous result is the equation of the constrain; the volume of the cylinder is given by:

$$V = \pi r^2 h$$

The composition of the functions yields:

$$V = \frac{\pi}{4}(400h - h^3)$$

...whose derivative is:

$$\frac{dV}{dh} = \frac{\pi}{4}(400 - 3h^2)$$

It implies that,

$$3h^2 = 400$$
$$h = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}}$$

Knowing the height at which the volume of the cylinder is maximum, we calculate the radius r and then the volume:

$$r^2 = \frac{400 - h^2}{4} = \frac{400 - (\frac{20}{\sqrt{3}})^2}{4} = \frac{200}{3}$$

The maximum possible volume of the cylinder is reached when $r^2 = \dfrac{200}{3}$ and $h = \dfrac{20}{\sqrt{3}}$:

$$V = \pi \left(\frac{200}{3}\right)\left(\frac{20}{\sqrt{3}}\right) = 2418\ cm^3$$

Miscellaneous exercises 8, page 126 No. 8 and 9

8. An arithmetic progression has first term a and common difference -1. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.

The sum of the first n terms: $S_n = \dfrac{n}{2}[2a + (n-1) - 1] = \dfrac{n}{2}(2a - n + 1)$.
The sum of the first $3n$ terms: $S_{3n} = \dfrac{3n}{2}[2a + (3n-1) - 1] = \dfrac{3n}{2}(2a - 3n + 1)$.
These two sums are equal to one another, therefore:

$$\frac{n}{2}(2a - n + 1) = \frac{3n}{2}(2a - 3n + 1).$$

Multiplying by 2, diving by n, we obtain:

$$2a - n + 1 = 6a - 9n + 3$$

$$9n - n + 1 - 3 = 6a - 2a$$

$$8n - 2 = 4a$$

$$4n - 1 = 2a \quad \text{then, a in terms of n is:} \quad a = \frac{4n-1}{2}$$

9. Find the sum of the arithmetic progression 1, 4, 7, 10, 13, 16,,1000. Every third term of the above progression is removed, i.e. 7, 16, etc. Find the sum of the remaining terms.

The given arithmetic progression has $d = 3$; the last term, l is 1000; then, the number of terms is given by: $l = a_1 + (n-1)d$ substituting in the values we have: $1000 = 1 + 3n - 3$; therefore $n = 334$; the sum of the progression is given by:

$$S_{1000} = \frac{n}{2}(a_1 + l) = \frac{334}{2}(1 + 1000) = 167167.$$

The removed sequence, *every third terms* is: 7, 16, ...; let's find the sum of such sequence, and subtract the resulting sum from the sum of the initial sequence. The removed sequence has difference $d = 9$ and a first term $a = 7$; the last term 997 (terms being removed minus 7 –the first term– are multiples of 9; which is given by the fact that each nth term is the sequence is given by $a_n = a_1 + (n-1)9$. Then, $997 = 7 + 9n - 9$ or, $n = 111$:

$$S_{111} = \frac{111}{2}(7 + 997) = 55722$$

Therefore, after removing each $3rd$ term of the sequence, the resulting sequence sum is: $S = 167167 - 55722 = 111445$.

Miscellaneous exercises 9, page 136 No. 17 and No. 28 page 137.

17. Find and simplify the term independent of x in the expansion of $\left(\dfrac{1}{2x} + x^3\right)^8$.

The term independent of x has the form kx^0, where k is a number and $x^0 = 1$; then, for $\left(\frac{1}{2x} + x^3\right)^8$ the term independent of x is given by: $\binom{8}{j}\left(\frac{1}{2x}\right)^{8-j}(x^3)^j = kx^0$; where the coefficient k is given by:

$$\binom{8}{j}\left(\frac{1}{2}\right)^{8-j} = k$$

While,

$$(x^{-1})^{8-j} \cdot x^{3j} = x^0$$
$$x^{j-8} \cdot x^{3j} = x^0$$
$$x^{j-8+3j} = x^0$$
$$j - 8 + 3j = 0, \quad \text{therefore} \quad 4j - 8 = 0; \quad \text{or} \quad j = 2.$$

The coefficient is:

$$\binom{8}{2}(2^{-1})^{8-2} = k$$
$$28 \cdot \frac{1}{64} = \frac{7}{16} = k$$

28. Prove that $\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$

Proving that:

$$\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$$

Right hand side: $\dfrac{(n+2)!}{(r+1)![(n+2)-(r+1)]!} = \dfrac{(n+2)!}{(r+1)!(n-r+1)!}$

Left hand side:

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{2n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

In order to simplify the previous result further, we have to consider that, according to the definition of the factorial,

i. $r! = r \cdot (r-1)!$ which can be written as $(r-1)! = \dfrac{r!}{r}$

ii. $r! = \dfrac{(r+1)r!}{(r+1)} = \dfrac{(r+1)!}{(r+1)}$

iii. Applying ii; then, $(n-r)! = \dfrac{(n-r+1)!}{(n-r+1)}$

iv. Again, applying ii twice; we have: $(n-r-1)! = \dfrac{(n-r)!}{(n-r)} = \dfrac{1}{(n-r)} \cdot \dfrac{(n-r)!}{1} = \dfrac{(n-r+1)!}{(n-r)(n-r+1)}$

Back to the left hand side,

$$= \frac{n!}{\frac{r!}{r}(n-r+1)!} + \frac{2n!}{\frac{(r+1)!}{(r+1)}\frac{(n-r+1)!}{(n-r+1)}} + \frac{n!}{(r-1)!\frac{(n-r+1)!}{(n-r)(n-r+1)}}$$

$$= \frac{n!\ r}{r!(n-r+1)!} + \frac{2n!\ (r+1)\ (n-r+1)}{(r+1)!(n-r+1)!} + \frac{n!\ (n-r)\ (n-r+1)}{(r-1)!(n-r+1)}$$

But, according to ii $r! = \dfrac{(r+1)!}{(r+1)}$ then, the first term becomes:

$\dfrac{n!\ r}{\frac{(r+1)!}{(r+1)}(n-r+1)!} = \dfrac{n!\ r\ (r+1)}{(r+1)!(n-r+1)!}$ thus, the left hand side is:

$$\frac{n!\ r\ (r+1)}{(r+1)!(n-r+1)!} + \frac{2n!\ (r+1)\ (n-r+1)}{(r+1)!(n-r+1)!} + \frac{n!\ (n-r)\ (n-r+1)}{(r-1)!(n-r+1)}$$

We have a common denominator in the left hand side; and, at the same time, that is the denominator in the right hand side; therefore, by adding algebraically the numerator in the left hand side we should get the right hand side numerator, that is $(n+2)!$. Let's work on the left hand side numerator:

$$n!r(r+1) + 2n!(r+1)(n-r+1) + n!(n-r)(n-r+1) = (n+2)!$$

$$n!\,[r(r+1) + 2(r+1)(n-r+1) + (n-r)(n-r+1)] = (n+2)!$$

$$n!\left[r^2 + r + 2(rn - r^2 + r + n - r + 1) + (n^2 - nr + n - nr + r^2 - r)\right] = (n+2)!$$

$$n!\left[r^2 + r + 2rn - 2r^2 + 2r + 2n - 2r + 2 + n^2 - 2nr + n + r^2 - r)\right] = (n+2)!$$

$$n![n^2 + 3n + 2] = (n+2)!$$

$$n!(n+1)(n+2) = (n+2)! \text{ which, by the definition of the factorial, is true!}$$

Miscellaneous exercises 11, page 173 No. 19

19. Given the function $f(x) = \dfrac{2x-4}{x}$ $x \in \mathbf{R}$ and $x \neq 0.$, find:

(a) $f^2(x)$ (b)$f^{-1}(x)$ (c)$f^4(x)$ (d)$f^{10}(x)$ (e) $f^{12}(x)$ (f)f^{82}

(The notation $f^2(x)$ represent the composite function ff; $f^3(x)$, represents fff, and so on.)

Note: By definition a function f that takes x to y, (f). An inverse function is another function that takes y, (f) back to x. This is, a function composed with its inverse function yields the original starting value: $f(f^{-1}) = x$ and $f^{-1}(f) = x$. Also, the repeated composition of a function with itself is called iterated function. A function iterated n times is sometimes denoted $f^n(x)$. And,trivial but so important, the composition of any given function, let's say, $g(x)$ with the function $f(x) = x$ yields $g(x)$.

(a) given $f(x) = \dfrac{2x-4}{x}$ the composition of the function with itself, ff,denoted as f^2 is given by: $f^2 = \dfrac{2\left[\frac{2x-4}{x}\right] - 4}{\left[\frac{2x-4}{x}\right]}$

$$= \frac{4x-8-4x}{x} \cdot \frac{x}{2x-4} = \frac{-8}{2x-4} = \frac{-4}{x-2} = \frac{4}{2-x}.$$

(b) $y = \dfrac{2x-4}{x}$ the inverse is given by: $yx = 2x - 4$, or:

$$4 = 2x - yx$$
$$4 = x(2-y)$$
$$\frac{4}{2-y} = x$$

We have obtained the inverse function: a formula that transform what we may called the old y into the new x.

Let's rename this function as the inverse of the original $f(x)$, as follows:

$$f^{-1} = \frac{4}{2-x}$$

(c) Answers to (a) and (b) prove that, for the given function, $ff = f^{-1}$; then fff which is the composition of f with ff is equivalent to the composition of f with its inverse f^{-1} which yield x (see note above). Therefore:

$$f^3 = fff = f^{-1}f = x$$

(d) Since $fff = x$ then f^4 is the composition of $f(x) = \dfrac{2x-4}{x}$ with x, which yields $f(x)$ itself.

(e) Whenever the composition of f with itself is a multiple of 3, the iteration (repeated composition of the function with itself) yields x. Let's visualize the pattern of the composition of f this way: $f^{12} = fff\ fff$
fff; that is, every three compositions of f with itself it yields x, and, of course, the composition of x with the x is x.

(f) f^{82}; since $82 = 3(27) + 1$ it means that, while $f^{81} = x$ (see previous result) f^{82} is given by the composition of $f(x)$ with x which yields $f(x) = \dfrac{2x-4}{x}$.

Miscellaneous exercises 9, page 137
32. Find an expression, in terms of n, for the coefficient of x in the expansion $(1+4x)+(1+4x)^2+(1+4x)^3+(1+4x)^4+...$
This is an infinite sum from $n = 1$ to n:

$\sum_{n=1}^{n}(1+4x)^n = (1+4x) + (1+4x)^2 + (1+4x)^3 + (1+4x)^4 + ...$

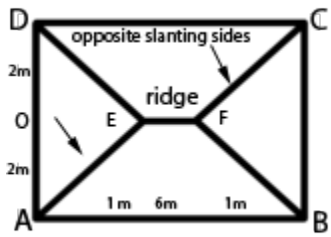
We have to find an expression for the coefficient of x in terms of n. The term that includes x, the second term in each of the expansions is given by $\binom{n}{1}1^n(4x)^1 = 4nx$. The infinite sum of these term becomes:

$\sum_{n=1}^{n} 4nx = 4 + 4(2)x + 4(3)x + ...$

Factoring out $4x$ we have the sum of the natural numbers from 1 to n; which is given by (see arithmetic sequences, chapter 8):
$4x\sum_{n=1}^{n} n = 4x \cdot \dfrac{n}{2}(n+1) = 2n(n+1)x.$

Exercise 13D, page 207
16. The roof of a house has a rectangular base of side 4 metres by 8 metres. The ridge line of the roof is 6 metres long, and centred 1 metre above the base of the roof. Calculate the acute angle between two opposite slanting edges of the roof.

Graph not drawn to scale: observe that AD is located along the x-axis; the origin, point O, $(0, 0, 0)$ divides AD into two equal segments. The line of the ridge would be on the y-axis:



The coordinates of interest are: $A = (2, 0, 0)$; $C = (-2, 8, 0)$; $E = (0, 1, 1)$; and $F = (0, 7, 1)$; Therefore,

$\vec{AE} = (-2, 1, 1)$; and $\vec{FC} = (-2, 1, 11)$ then, $\vec{AE} \cdot \vec{FC} = (4 + 1 - 1) = 4$ while $|\vec{AE}| = \sqrt{6}$; $|\vec{FC}| = \sqrt{6}$,

$\theta = \cos^{-1}\left(\dfrac{4}{\sqrt{6}\,\sqrt{6}}\right) = 48.18^\circ = 48.2^\circ$