Practice 7

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Suggested solutions for exercises 6D —number 1 to 13— page 86 of Pure Mathematics 1 by Hugh Neil and Douglas Qualing.

Note about differentiating functions:

i. The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.
ii. The derivative of $f(x) = kx$ is $f'(x) = k$.
Since the derivative of a function at a point is the gradient of the tangent line at that point, in the case of $f(x) = kx$ this is the equation of a straight line whose gradient is constant and equal to k. Besides, applying the nx^{n-1} rule kx which is read as kx^1 yields $k(1)x^{1-1} = kx^0 = k$.
ii. The derivative of $f(x) = k$ is f'(x)=0. This result can be explained by the fact that the equation of $f(x) = k$ describes a horizontal line whose gradient, at every point, is constant and equal to zero. Also, by applying the power rule to $f(x) = kx^0$ we get $f'(x) = k(0)x^{-1} = 0$.

1.

a)$f(x) = x^3 + 2x^2$ the derivative of f(x) is: $f'(x) = 3x^2 + 4x$

b)$f(x) = 1 - 2x^3 + 3x^2$ whose derivative is: $f'(x) = -6x^2 + 6x$

c)$f(x) = x^3 - 6x^2 + 11x - 6$ then $f'(x) = 3x^2 - 12x + 11$

d)$f(x) = 2x^3 - 3x^2 + x$ the derivative is: $f'(x) = 6x^2 - 6x + 1$

e)$f(x) = 2x^2(1 - 3x^2) = 2x^2 - 6x^4$ whose derivative is: $f'(x) = 4x - 24x^3$

f)$f(x) = (1 - x)(1 + x + x^2) = 1 - x^3$ whose derivative is given by: $f'(x) = -3x^2$

2.

a) $f(x) = 2x - x^3$ $f(-2) = 2 - 3(-2)^2 = -10.$

b) $f(x) = 2x - x^2$ $f(-2) = 2 - 3(-2) = 6.$

c) $f(x) = 1 - 2x - 3x^2 + 4x^3$ $f(-2) = -2 - 6(-2) + 12(-2)^2 = 58.$

d) $f(x) = 2 - x$ $f'(x) = -1$ $f'(-2) = -1.$

e) $f(x) = x(1 + x)^2$ $f(-2) = 2(-2) + 3(-2)^2 = 8.$

f) $f(x) = (1 + x)(1 - x + x^2) = 1 + x^3$ $f'(-2) = 3(-2)^2 = 12.$

3.

a) $f(x) = x^3 \quad f'(x) = 3x^2 \qquad 3x^2 = 12$ therefore $x = \pm 2$.

b) $f(x) = x^3 - x^2 \quad f'(x) = 3x^2 - 2x \qquad 3x^2 - 2x = 8$

$3x^2 - 2x - 8 = 0$ it factorizes into: $(x-2)(3x+4) = 0$ then, $x = 2$ *or* $x = -\frac{4}{3}$.

c) $f(x) = 3x - 3x^2 + x^3 \quad f'(x) = 3 - 6x + 3x^2 \qquad 3 - 6x + 3x^2 = 108$

$3x^2 - 6x - 105 = 0$ this is equivalent to: $x^2 - 2x - 35 = 0$ it factorizes into: $(x-7)(x+5) = 0$ therefore $x = 7$ *or* $x = -5$.

d) $f(x) = x^3 - 3x^2 + 2x \quad f'(x) = 3x^2 - 6x + 2 \qquad 3x^2 - 6x + 2 = -1$
$3x^2 - 6x + 3 = 0$ which is equivalent to: $x^2 - 2x + 1 = 0 \; = \; (x-1)^2 = 0$. Therefore, $x = 1$.

e) $f(x) = x(1+x)^2 \; = x + 2x^2 + x^3 \quad f'(x) = 1 + 4x + 3x^2 = 0$ which in factored form becomes $(x+1)(3x+1) = 0$ hence, $x = -1$ or $x = -\frac{1}{3}$.

f) $f(x) = x(1-x)(1+x) = x(1-x^2) = x - x^3 \; f'(x) = 1 - 3x^2 = 2$. Therefore, $3x^2 = -1$ This equation has no Real number solutions, since the square root of a negative number has no meaning in the Real number system.

4.

a) $f(x) = 2\sqrt{x} = 2x^{\frac{1}{2}} \quad f'(x) = 2(\frac{1}{2})(x^{-\frac{1}{2}}) = \dfrac{1}{\sqrt{x}}$.

b) $f(x) = (1+\sqrt{x})^2 = 1 + 2\sqrt{x} + x = 1 + 2x^{\frac{1}{2}} + x \qquad f'(x) = x^{-\frac{1}{2}} + 1 = \dfrac{1}{\sqrt{x}} + 1$.

c) $f(x) = x - \frac{1}{2}\sqrt{x} \quad f'(x) = 1 - \frac{1}{4}x^{-\frac{1}{2}} = 1 - \dfrac{1}{4\sqrt{x}}$.

d) $f(x) = x\left(1 - \frac{1}{\sqrt{x}}\right)^2 = x\left(1 - \frac{2}{\sqrt{x}} + \frac{1}{x}\right) = \left(x - \frac{2x}{\sqrt{x}} + 1\right) = (x - 2\sqrt{x} + 1) \quad f'(x) = 1 - \frac{1}{\sqrt{x}}$.

e) $f(x) = x - \frac{1}{x} = x - x^{-1} \quad f'(x) = 1 - [-x^{-2}] = 1 + \dfrac{1}{x^2}$.

f) $f(x) = \dfrac{x^3 + x^2 + 1}{x} \quad f(x) = x^2 + x + x^{-1} \quad f'(x) = 2x + 1 - x^{-2} \quad f'(x) = 2x + 1 - \dfrac{1}{x^{-2}}$.

g) $f(x) = \dfrac{(x+1)(x+2)}{x} = \dfrac{x^2 + 2x + x + 2}{x} = x + 3 + 2x^{-1} \quad f'(x) = 1 - \dfrac{2}{x^2}$.

h)$f(x) = \left(\dfrac{\sqrt{x} + x}{\sqrt{x}}\right)^2 = \left(\dfrac{\sqrt{x}}{\sqrt{x}} + \dfrac{x}{\sqrt{x}}\right)^2 = \left(1 + \sqrt{x}\right)^2$ same as 4 b), therefore the answer is: $f'(x) = \dfrac{1}{\sqrt{x}} + 1$.

5.

We are asked to find the equation of the tangent line to a given curve ($y = x^3 + x$) at a given x-coordinate: $x = -1$.
Calculate the y-coordinate: $f(-1) = (-1)^3 + (-1) = -2$. So the point is $(-1, -2)$.
Find the value of the derivative at $x = -1$: this is the gradient of the tangent line: $f'(-1) = 3(-1)^2 + 1 = 4$.
Equation of the tangent line: $y - (-2) = 4[x - (-1)]$ or $y + 2 = 4x + 4$, therefore: $y = 4x + 2$.

6.

We are told that the line with equation $y = x - 2$ is tangent to $f(x) = 4x - x^3$. We need to find the other tangent parallel to $y = x - 2$.

Since the lines are parallel to each other, they have the same gradient; that is, $m = 1$ (this is the gradient of ($y = x - 2$). So, at what x-coordinate the derivative is equal to 1?

Solve: $f'(x) = 4 - 3x^2 = 1$ or $4 - 3x^2 = 1$, $3x^2 = 3$, therefore: $x \pm 1$.
Calculate $f(-1) = 4(-1) - (-1)^3 = -3$ point $(-1, -3)$ and $f(1) = 4(1) - (1)^3 = 3$, point $(1, 3)$. For a gradient equal to 1, at point (-1, -3) the equation of the tangent touching that point is $y + 3 = 1(x + 1)$ or $y = x - 2$. This is the equation of the line given in the question.
The other line should has equation whose gradient is also equal to 1, at point (1,3):
$y - 3 = 1(x - 1)$ or $y = x + 2$.

7.

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$; $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $f'(4) = \frac{1}{2}(4)^{-\frac{1}{2}} = \dfrac{1}{4}$. Therefore, the gradient of the tangent line is equal to $\dfrac{1}{4}$.
The equation is: $y - 2 = \dfrac{1}{4}(x - 4)$ or $y = \dfrac{1}{4}x + 1$ which is equivalent to: $4y = x + 4$.

8.

Same type of question as number 7:

$fx) = \frac{1}{x} = x^{-1}$. $f'(x) = -x^{-2}$; evaluated at $x = 2$: $f'(2) = -2^{-2} = -\frac{1}{4}$; equation of the tangent line to the point $(2, \frac{1}{2})$ is: $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$ or $4y - 2 = -x + 2$ equivalent to: $4y = -x + 4$.

9.

Finding the equation of the normal to a point. The normal is perpendicular to the tangent line; therefore, its gradient is the negative reciprocal of the tangent line's gradient. So, firstly, we calculate the gradient of the tangent to the curve $(y = x + \frac{1}{x})$ at the given point: $(1, 2)$.

$f'(x) = 1 - \frac{1}{x^2}$ $f'(1) = 1 - \frac{1}{1^2} = 0$

The gradient of the normal is the negative reciprocal of zero; this quantity is undefined. An undefined gradient correspond to vertical line. Therefore, the vertical line, passing through point (1,2) has equation $x = 1$.

10.

The graphs of $f(x) = x^2 - 2x$ and $f(x) = x^3 - 3x^2 - 2x$ pass through the origin [for both functions it is true that $f(0) = 0$]. We need to prove that the functions share the same tangent lines at (0,0).

For the first function, the derivative is: $f'(x) = 2x - 2$, so the gradient of the tangent line at $x = 0$ is given by: $f'(0) = -2$;

For the second function, the derivative is: $f'(x) = 3x^2 - 6x - 2$, so the gradient of the tangent line at $x = 0$ is given by: $f'(0) = -2$;

Therefore, for both function we have the same point (0, 0) and the the gradient of the tangent line at that point is also the same for both curves: necessarily, the functions same tangent lines at the origin and the equation of such a line is: $y - 0 = -2(x - 0)$ or $y = -2x$.

11.

Tangent to the curve at the point where it crosses the y-axis ($x = 0$). $f(x) = x^3 - 3x^2 - 2x - 6$. $f(0) = -6$. We need to find the tangent to the curve at $(0, -6)$.

$f'(x) = 3x^2 - 6x - 2$. $f'(0) = -2$. Equation of the tangent: $y - (-6) = -2(x - 0)$ or $y + 6 = -2x$, which is equivalent to $y = -2x - 6$.

12.

The curve has equation $y = x(x - a)(x + a)$ where "a" is a constant crosses the x-axis $(y = 0)$ at $x = 0$, $x = a$ and $x = -a$. That is, the points are: $(0, 0)$; $(a, 0)$ and $(-a, 0)$.

The equation of the curve in factored form is equivalent to: $y = x(x^2 - a^a)$ or $y = x^3 - a^x$. Its derivative is: $y' = 3x^2 - a^2$.

* Equation of tangent line at $(0, 0)$ has gradient $f'(0) = -a^2$: $y - 0 = -a^2(x - 0)$ or $y = -a^2x$.

* Equation of tangent line at $(a, 0)$ has gradient $f'(a) = 2a^2$: $y - 0 = 2a^2(x - a)$ or $y = 2a^2x - 2a^3$.

* Equation of tangent line at $(-a, 0)$ has gradient $f'(-a) = 2a^2$: $y - 0 = 2a^2(x + a)$ or $y = 2a^2x + 2a^3$.

13.

Coordinates of intersection of the points of tangents to the curve $y = x^2$ at points at which the curve $y = x^2$ meets the line $y = x + 2$.

The curve and the line meet at:

$x^2 = x + 2$. or $x^2 - x - 2 = 0$ in factored form becomes: $(x - 2)(x + 1) = 0$. That is, they meet at $x = 2$ and $x = -1$. Finding the y-coordinates in either of the functions we obtain: $(2, 4)$ and $(-1, 1)$.

Equation of the tangent line at $(2, 4)$:
$f'(x) = 2x$, $f'(2) = 4$; the equation of the tangent is: $y - 4 = 4(x - 2)$. or $y = 4x + 4$.

Equation of the tangent line at $(-1, 1)$:
$f'(x) = 2x$, $f'(-1) = -2$; the equation of the tangent is: $y - 1 = -2(x + 1)$. or $y = -2x - 14$.

The two tangent lines meet at:

$$4x - 4 = -2x - 1$$
$$6x = 3$$
$$x = \frac{1}{2}$$
$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) - 4 = -2$$

The two tangent lines meet at $\left(\frac{1}{2}, -2\right)$.