Practice 1

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Suggested solutions for miscellaneous exercises 1, page 15 of Pure Mathematics 1 by Hugh Neil and Douglas Qualing.

1.

We have to prove that points (-2, 5), (1, 3) and (5, 9) determine a right triangle:
Method 1: Label (-2, 5) as A, (1, 3) as B and (5, 9) as C. Then the gradient of the line (side of triangle) AB $m = \frac{(y_2-y_1)}{(x_2-x_1)} = -\frac{3}{2}$. Gradient of BC, $m = \frac{3}{2}$; since the gradients are the negative reciprocal of one another, side (line) AB⊥BC.
Method 2: Converse of Pythagorean Theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.
Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
The distance between (-2, 5) and (1, 3) = $\sqrt{13}$; Distance between (1, 3) and (5, 9) = $\sqrt{52}$ and the distance between (-2, 5)and (5, 9) = $\sqrt{65}$
Since $(\sqrt{65})^2 = (\sqrt{13})^2 + (\sqrt{52})^2$ then, by the converse of Pythagorean Theorem the triangle is a right triangle.

2.

The coordinates of the point at which lines $2x + y = 3$ and $3x + 5y - 1 = 0$ meet, are:
Solving for y and then set both equations equal to each other:

$$3 - 2x = \frac{1 - 3x}{5}$$

Multiply both sides of the equation by 5, and combine like terms: $7x = 14$ therefore $x = 2$
Since this is the value of x at which the lines meet, the y coordinate can be found by substituting $x = 2$ into either equation: $y = 3 - 2x$, or $y = 3 - 2(2) = -1$ Answer: The coordinates of the point is (2, -1).

3.

a) The triangle is formed by A (-1, 3), B (5, 7) and C (0, 8). We have to prove that angle ACB is a right angle. In that case line (side) AC⊥CB. By following method 1 in question 1, the gradient of AC, $m = \frac{5}{1}$ similarly, gradient of CB, $m = -\frac{1}{5}$ so, since they are the negative reciprocal of one another, side (line) AC⊥CB or ACB is a right angle.
b) The line through B (5, 7) parallel to AC – since the lines are parallel the gradients are equal– $m_{AC} = 5$ the equation of the line is $y - 7 = 5(x - 5)$ or $y = 5x - 18$ The line cuts the x-axis at y=0; solving for x, $0 = 5x - 18$ or $x = \frac{18}{5} = 3.6$. The coordinates of the point are (3.6, 0).

4.

a) A (7, 2) and C (1, 4) are two vertices of a square ABCD. In order to find the equation of the diagonal BD let's calculate the gradient of AC – labeled m_1 –, then, considering diagonals of the square are perpendicular to each other, the gradient of BD is $-\frac{1}{m_1}$. Diagonals of squares bisect each other; therefore the mid point of AC lies on BD. Knowing the gradient and the coordinates of a point we are able to find the equation of line BD as follows:

$$m_1 = \frac{4-2}{1-7} = -\frac{1}{3}$$

$$m_{BD} = 3$$

Midpoint of AC: $(\frac{8}{2}, \frac{6}{2}) = (4, 3)$ Equation of BD, $y - 3 = 3(x - 4)$ or $y = 3x - 9$

b) In order to find the coordinates of B and D, let's calculate the distance from the midpoint to C (d_1); that's would be the same distance from the midpoint to both B and D.

$$d_1 = \sqrt{(3)^2 + (-1)^2} = \sqrt{10}$$

Therefore,

$$\sqrt{10} = \sqrt{(4-x)^2 + (3-y)^2}$$
$$0 = 10x^2 - 80x + 150$$
$$0 = x^2 - 8 + 15 = (x-3)(x-5)$$
$$x_1 = 3; x_2 = 5$$

Substituting those values in the equation of the line [diagonal BD found in a)] yields $y_1 = 0$ and $y_2 = 6$ Therefore the coordinates of B and D are (3, 0) and (5, 6).

5.

a) In order to show that quadrilateral ABCD formed by the points A(-3, 2), B(4, 3), C(9, -2) and D (2, -3) all four sides are of equal lengths, calculate the distances AB, BC etc again by using the distance formula. The results is $d = \sqrt{50}$ for all four sides.
b) In order to prove that this geometric figure is not a square, let's calculate the gradients of two adjacent sides: $m_{AD} = -\frac{5}{5} = -1$ and $m_{DC} = \frac{1}{7}$ Since those sides are not perpendicular to one another, they don't form a right angle. The quadrilateral is a rhombus, but not a square.

6.

a) The given line, $3x + 4y = 16$ has a gradient of $m_1 = -\frac{3}{4}$ (for lines written in the form $ax + by = c$ the gradient is $= -\frac{a}{b}$). Since l_2 is $\perp$ to l_1, $l_2 = -\frac{1}{m_1}$ or $l_2 = \frac{4}{3}$ in this case. l_2 passes though P(7, 5) the equation of l_2 is $y - 5 = \frac{4}{3}(x - 7)$ or $4x - 3y = 13$

b)In order to find the point at which both lines meet, we simply solve this system of two equations. Let's solve for y in both equations, and equate the results in order to find the x coordinate of the point:

$$\frac{-3x + 16}{4} = \frac{4x - 13}{3}$$
$$-9x + 48 = 16x - 52$$
$$100 = 25x$$
$$x = 4$$

and, solving for y in one of the equations we find that $y = 1$; this is, the point of intersection of the lines is (4, 1).
c) The perpendicular distance from P to l_1 is just the distance form (7, 5) to (4, 1) the point at which both lines meet: $d = \sqrt{(3)^2 + (4)^2}$, $d = 5$

7.

We need to prove that a triangle with vertices (-2, 8), (3, 20) and (11, 8) is isosceles; this is, two sides are of equal lengths; and a third side, which we called the base, has a different length from the other two. By the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ side from (-2, 8) to (3, 20), $d = 13$, side from (-2, 8) to (11, 8), $d = 13$, and the distance from (3, 20) to (11, 8), $d = \sqrt{208}$. Therefore, since two sides are of equal size, and a third is of different size, the triangle is an isosceles triangle.
c) In order to calculate the area of the triangle $A = \frac{1}{2}b.h$ we need to calculate the height first. The base, $b = \sqrt{208}$ is known.The height is the perpendicular line from the vertex (-2, 8) to the base. The base is the line segment that connect vertices (3, 20) and (11, 8). The gradient of that line is $m = -\frac{3}{2}$ the equation of the line is $y - 20 = -\frac{3}{2}(x - 3)$; the equation of the line segment (height, perpendicular to the base, has a gradient $m = \frac{2}{3}$ (negative reciprocal of the base's gradient); therefore the equation of the height would be $y - 8 = \frac{2}{3}(x + 2)$. Solving for y, and equating both equations we are able to find the point at which the height meet the base. This is:

$$\frac{2}{3}x + \frac{28}{3} = -\frac{3}{2}x + \frac{49}{2}$$

$$4x + 56 = -9x + 147$$

$$13x = 91$$

$$x = 7$$

Substituting $x = 7$ into one of the line segment equations (sides) yields $y = 14$. Then the height is the distance from point (-2, 8) to (7, 14); $h = \sqrt{(9)^2 + (6)^2}$ or $h = \sqrt{117}$ The area of the triangle is $A = \frac{1}{2}(\sqrt{208})(\sqrt{117}) = 78$.

8.

The equations of three lines are given. We need to find the points of intersection of the lines, this is, the vertices of the triangle. The equations are: $y = x$, $7y = 2x$, and $4x + y = 60$. Solving for y (when necessary) and substituting one into another, $y = x$ meets $7y = 2x$ at $x = 0$; $y = x$ meets $4x + y = 60$ at $x = 12$ and $7y = 2x$ meets $4x + y = 60$ at $x = 14$; by plugging in the x values into one of the equation in each case, the coordinates of the vertices of the triangle are: (0, 0); (12, 12) and (14, 4).

9.

We need to find the equation of the line through (1, 3) parallel to $2x + 7y = 5$. Since the lines are parallel, the gradient of the lines are the same. The given equation has a gradient $m = -\frac{a}{b}$ (following the form $ax + by = c$, solving for y: $y = -\frac{ax}{b} + \frac{c}{a}$.) Therefore, a line trough (1, 3) whose gradient is $-\frac{2}{7}$ has equation $y - 3 = -\frac{2}{7}(x - 1)$ or $7y - 21 = -2x + 2$ which is equivalent to $2x + 7y = 23$.

10.

We are asked to find the equation of the perpendicular bisector of the line joining (2, -5) and (-4, 3). Firstly, let's calculate the gradient of the line connecting those two points: $m = -\frac{8}{6} = -\frac{4}{3}$ since the bisector is perpendicular to this line, its gradient will be $m = \frac{3}{4}$, and, a bisector cuts the line at the midpoint, which is in this case $(-\frac{2}{2}, -\frac{2}{2}) = (-1, -1)$ Knowing the coordinates of a point and the gradient of the line, the equation is $y+1 = \frac{3}{4}(x+1)$ or $4y + 4 = 3x + 3$ in $ax + by = c$ format the answer is $3x - 4y = 1$.

11.

Points A(1, 2), B(3, 5), C(6,6) and D form a parallelogram. We are asked to find the mid-point of AC and the coordinates of D. The mid-point coordinates are: $(\frac{7}{2}, \frac{8}{2}) = (\frac{7}{2}, 4)$. Since the diagonals bisect each other $(\frac{7}{2}, 4)$ is also the mid-point of BD. Applying the mid-point formula, and considering D coordinates (x, y) and B (3, 5) we have: $\frac{3+x}{2} = \frac{7}{2}$ or $x = 4$. For the y coordinate we have: $\frac{5+y}{2} = 4$ or $y = 3$, this is, the coordinates of D are (4, 3).

12.

Point P is the foot of the perpendicular from the point A(0, 3) to the line $y = 3x$.
a) The equation of the line AP is determined by a gradient which is the negative reciprocal of 3; this is $-\frac{1}{3}$. The line whose gradient is such and passes through (0, 3) has equation $y - 3 = -\frac{1}{3}(x - 0)$ or $x + 3y = 9$.
b) The coordinates of point P are determined by the point at which both lines meet: $y = 3x$ and $x+3y = 9$. Plugging in the first equation into the second yields $x + 9x = 9$ or $x = 0.9$; therefore $y = 2.7$. Answer: P coordinates are (0.9, 2.7).
The perpendicular distance from A(0, 3) to the line, point P (0.9, 2.7) is given by $d = \sqrt{(0.9)^2 + (0.3)^2} = \sqrt{0.90}$.