
5.

Note: in the process of composition the output of one function becomes the input of another. Repeated composition of such a function with itself is called iterated function. A function iterated \( n \) times is sometimes denoted \( f^n(x) \).

And, trivial but so important, the composition of any given function, let’s say, \( g \) with the function \( f(x) = x \) yields \( g(x) \).

(a) \( f(x) = x^\frac{1}{3} + 10 \) therefore composition of \( f \) with \( f \) is given by: 

\[
ff : x \mapsto (x^\frac{1}{3} + 10)^\frac{1}{3} + 10 = ff : (-8) \mapsto ((-2)^\frac{1}{3} + 10)^\frac{1}{3} + 10 \]

\[
= ff : (-8) \mapsto (-2 + 10)^\frac{1}{3} + 10 = ff : (-8) \mapsto (8)^\frac{1}{3} + 10 = ff : (-8) \mapsto (2)^\frac{1}{3} + 10 = 12;
\]

(b) Inverse function of \( f \):

\[
y = x^\frac{1}{3} + 10
\]

\[
(y - 10)^3 = x
\]

\[
f^{-1} : x \mapsto (x - 10)^3
\]

\[
f^{-1} : 13 \mapsto (13 - 10)^3 = 27.
\]

16.

\( f : \mapsto 2x + 7 \) and \( g : \mapsto x^3 - 1 \); then,

(a) \( f^{-1} \)

\[
y = 2x + 7
\]

\[
y - 7 = 2x
\]

\[
y - 7 = \frac{x}{2}
\]

Therefore, we rename this function as the inverse of the original:

\[
f^{-1} = \frac{x - 7}{2} = \frac{1}{2}(x - 7).
\]

(b) \( g^{-1} \)

\[
y = x^3 - 1
\]

\[
y + 1 = x^3
\]

\[
\sqrt[3]{y + 1} = x
\]
The inverse is of $g$ is $g^{-1} = \sqrt[3]{x + 1}$

c) The composition of $f^{-1}$ with $g^{-1}$:

$$g^{-1}f^{-1} = \sqrt[3]{\frac{x - 7}{2} + 1}$$
$$g^{-1}f^{-1} = \sqrt[3]{\frac{x - 5}{2}}$$
$$g^{-1}f^{-1} = \frac{1}{2}(x - 5)$$

(d) $f^{-1}g^{-1}$ This is the composition of $g^{-1}$ with $f^{-1}$:

$$f^{-1}g^{-1} = \frac{\sqrt[3]{x + 1} - 7}{2} = \frac{1}{2}(\sqrt[3]{x + 1} - 7)$$

(e) $fg = 2(x^3 - 1) + 7 = 2x^3 - 2 + 7 = 2x^3 + 5$.

(f) $gf = (2x + 7)^3 - 1$.

(g) $(fg)^{-1}$ denotes the inverse of the composition of $fg$ — see item (g):

$$y = 2x^3 + 5$$

Then, solving for $x$, we have:

$$\frac{1}{2}(y - 5) = x^3$$
$$\sqrt[3]{\frac{1}{2}(y - 5)} = x$$

Therefore the inverse of $(fg)$ is given by

$$(fg)^{-1} = \sqrt[3]{\frac{1}{2}(x - 5)}$$

(h) $(gf)^{-1}$ denotes the inverse of the composition of $gf$:

Per item (f), $(fg) = (2x + 7)^3 - 1$, then, solving or x:

$$y = (2x + 7)^3 - 1$$
$$y + 1 = (2x + 7)^3$$
$$\sqrt[3]{y + 1} = 2x + 7$$
$$\frac{1}{2}(\sqrt[3]{y + 1} - 7) = x$$

$$(gf)^{-1} = \frac{1}{2}(\sqrt[3]{x + 1} - 7)$$

17.

$f : \mapsto 10 - x$, for all real numbers. Let's consider the composition of $f$ with itself (iteration):

$$ff : \mapsto 10 - (10 - x) = x$$
That is, the composition of $f$ —or iteration of $f$ — an odd number of times is given by: $f^{odd} = 10 - x$ while $f^{even} = x$; then:

(a) $f(7) = 10 - 7 = 3$
(b) $f^2(7) = 7$
(c) $f^{15}(7) = 10 - 7 = 3$
(d) $f^{100}(7) = 7$

19.

Note: By definition a function $f$ that takes $x$ to $y$, $(f)$. An inverse function is another function that takes $y$, $(f^{-1})$ back to $x$. This is, a function composed with its inverse function yields the original starting value: $f(f^{-1}) = x$ and $f^{-1}(f) = x$.

(a) given $f(x) = \frac{2x - 4}{x}$ the composition of the function with itself, $ff$, denoted as $f^2$ is given by: $f^2 = \frac{2}{\frac{2x - 4}{x}} - 4$

(b) $y = \frac{2x - 4}{x}$ the inverse is given by: $yx = 2x - 4$, or:

\[
\begin{align*}
4 &= 2x - yx \\
4 &= x(2-y) \\
\frac{4}{2-y} &= x
\end{align*}
\]

we have obtained the inverse function: a formula that transform what we may called the old $y$ into the new $x$

Let’s rename this function as the inverse of the original $f(x)$, as follows:

\[
f^{-1} = \frac{4}{2-x}
\]

(c) Answers to (a) and (b) prove that, for the given function, $ff = f^{-1}$; then $fff$ which is the composition of $f$ with $ff$ is equivalent to the composition of $f$ with its inverse $f^{-1}$ which yield $x$ (see note above). Therefore:

\[
f^3 = fff = f^{-1}f = x
\]

(d) Since $fff = x$ then $f^4$ is the composition of $f(x) = \frac{2x - 4}{x}$ with $x$, which yields $f(x)$ itself.

(e) Whenever the composition of $f$ with itself is a multiple of 3, the iteration (repeated composition of the function with itself) yields $x$. Let’s approach the pattern of the composition of $f$ this way: $f^{12} = fff fff fff$, that is, every three compositions of $f$ with itself it yields $x$, and, of course, the composition of $x$ with the $x$ is $x$.

(f) $f^{82}$; since $82 = 3(27) + 1$ it means that, while $f^{81} = x$ (see previous result) $f^{82}$ is given by the composition of $f(x)$ with $x$ which, of course, yields $f(x) = \frac{2x - 4}{x}$.
(a) A function is self-inverse if \( ff = x \); for \( f : \frac{x + a}{x - 1} \) the composition \( ff \) yields:

\[
ff = \frac{x + a}{x - 1} + a
\]

\[
ff = \frac{x + a}{x - 1} - 1
\]

\[
ff = \frac{x + a + a(x - 1)}{x + a - (x - 1)}
\]

\[
ff = \frac{x + 1 + a}{x - 1} \cdot \frac{x - 1}{a + a - x + 1}
\]

\[
ff = \frac{x + ax}{x - 1} \cdot \frac{x - 1}{a + 1}
\]

\[
ff = \frac{x(1 + a)}{1 + a} = x.
\]

Therefore the function \( f : \frac{x + a}{x - 1} \) is a self-inverse, since \( f(f^{-1}) = x \) and the composition of the function with itself also yields \( x \).