Practice 15

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Suggested solutions for Miscellaneous exercise 11 page 172-173 No.1 thru No.20, from Pure Mathematics 1 by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

5.

Note: in the process of composition the output of one function becomes the input of another. Repeated composition of such a function with itself is called iterated function. A function iterated n times is sometimes denoted $f^n(x)$. And,trivial but so important, the composition of any given function, let's say, $g(x)$ with the function $f(x) = x$ yields $g(x)$.

(a) $f(x) = x^{\frac{1}{3}} + 10$ therefore composition of f with f is given by: $ff : x \mapsto (x^{\frac{1}{3}} + 10)^{\frac{1}{3}} + 10$ then ff(-8)= $ff : (-8) \mapsto ((-2^3)^{\frac{1}{3}} + 10)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (-2 + 10)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (8)^{\frac{1}{3}} + 10 = ff : (-8) \mapsto (2^3)^{\frac{1}{3}} + 10$ $= 12$;

(b) Inverse function of f:

$$y = x^{\frac{1}{3}} + 10$$

$$(y - 10)^3 = x$$

$$f^{-1} : x \mapsto (x - 10)^3$$

$$f^{-1} : 13 \mapsto (13 - 10)^3 = 27.$$

16.

$f : \mapsto 2x + 7$ and $g : \mapsto x^3 - 1$; then,

(a) f^{-1}

$$y = 2x + 7$$

$$y - 7 = 2x$$

$$\frac{y - 7}{2} = x,$$

Therefore, we rename this function as the inverse of the original:

$f^{-1} = \dfrac{x - 7}{2} = \dfrac{1}{2}(x - 7).$

(b) g^{-1}

$$y = x^3 - 1$$

$$y + 1 = x^3$$

$$\sqrt[3]{y + 1} = x$$

The inverse is of g is $g^{-1} = \sqrt[3]{x+1}$

c) The composition of f^{-1} with g^{-1}:

$$g^{-1}f^{-1} = \sqrt[3]{\frac{x-7}{2}+1}$$
$$g^{-1}f^{-1} = \sqrt[3]{\frac{x}{2}-\frac{7}{2}+1}$$
$$g^{-1}f^{-1} = \sqrt[3]{\frac{x}{2}-\frac{5}{2}}$$
$$g^{-1}f^{-1} = \sqrt[3]{\frac{1}{2}(x-5)}$$

(d) $f^{-1}g^{-1}$ This is the composition of g^{-1} with f^{-1} :

$$f^{-1}g^{-1} = \frac{\sqrt[3]{x+1}-7}{2} = \frac{1}{2}(\sqrt[3]{x+1}-7)$$

(e)$fg = 2(x^3-1)+7 = 2x^3-2+7 = 2x^3+5.$

(f)$gf = (2x+7)^3-1.$

(g) $(fg)^{-1}$ denotes the inverse of the composition of $fg :$ —see item (g):

$$y = 2x^3+5$$

Then, solving for x, we have:

$$\frac{1}{2}(y-5) = x^3$$
$$\sqrt[3]{\frac{1}{2}(y-5)} = x$$

Therefore the inverse of (fg) is given by

$$(fg)^{-1} = \sqrt[3]{\frac{1}{2}(x-5)}$$

(h)$(gf)^{-1}$ denotes the inverse of the composition of gf :

Per item (f), $(gf) = (2x+7)^3-1,$ then, solving or x:

$$y = (2x+7)^3-1$$
$$y+1 = (2x+7)^3$$
$$\sqrt[3]{y+1} = 2x+7$$
$$\frac{1}{2}\left(\sqrt[3]{y+1}-7\right) = x$$
$$(gf)^{-1} = \frac{1}{2}\left(\sqrt[3]{x+1}-7\right)$$

17.

$f :\mapsto 10-x$, for all real numbers. Let's consider the composition of f with itself (iteration):

$$ff :\mapsto 10-(10-x) = x$$

$$fff \texttt{ or } f^3 = 10 - x$$
$$f^4 : \mapsto 10 - (10 - x) = x$$
$$f^5 = 10 - x$$

That is, the composition of f —or iteration of f — an odd number of times is given by: $f^{odd} = 10 - x$ while $f^{even} = x$; then:

(a)$f(7) = 10 - 7 = 3$
(b)$f^2(7) = 7$
(c)$f^{15}(7) = 10 - 7 = 3$
(d)$f^{100}(7) = 7$

19.

Note: By definition a function f that takes x to y, (f). An inverse function is another function that takes y, (f) back to x. This is, a function composed with its inverse function yields the original starting value: $f(f^{-1}) = x$ and $f^{-1}(f) = x$.

(a) given $f(x) = \dfrac{2x-4}{x}$ the composition of the function with itself, ff,denoted as f^2 is given by: $f^2 = \dfrac{2\left[\dfrac{2x-4}{x}\right] - 4}{\left[\dfrac{2x-4}{x}\right]}$

$= \dfrac{4x - 8 - 4x}{x} \cdot \dfrac{x}{2x-4} = \dfrac{-8}{2x-4} = \dfrac{-4}{x-2} = \dfrac{4}{2-x}$.

(b) $y = \dfrac{2x-4}{x}$ the inverse is given by: $yx = 2x - 4$, or:

$$4 = 2x - yx$$
$$4 = x(2-y)$$
$$\frac{4}{2-y} = x$$

we have obtained the inverse function: a formula that transform what we may called the old y into the new x

Let's rename this function as the inverse of the original $f(x)$, as follows:

$$f^{-1} = \frac{4}{2-x}$$

(c) Answers to (a) and (b) prove that, for the given function, $ff = f^{-1}$; then fff which is the composition of f with ff is equivalent to the composition of f with its inverse f^{-1} which yield x (see note above). Therefore:

$$f^3 = fff = f^{-1}f = x$$

(d) Since $fff = x$ then f^4 is the composition of $f(x) = \dfrac{2x-4}{x}$ with x, which yields $f(x)$ itself.

(e) Whenever the composition of f with itself is a multiple of 3, the iteration (repeated composition of the function with itself) yields x. Let's approach the pattern of the composition of f this way: $f^{12} = fff\ fff$
fff, that is, every three compositions of f with itself it yields x, and, of course, the composition of x with the x is x.

(f) f^{82}; since $82 = 3(27) + 1$ it means that, while $f^{81} = x$ (see previous result) f^{82} is given by the composition of $f(x)$ with x which, of course, yields $f(x) = \dfrac{2x-4}{x}$.

20.

(a) A function is self-inverse if $ff = x$; for $f :\mapsto \dfrac{x+a}{x-1}$ the composition ff yields:

$$ff = \frac{\dfrac{x+a}{x-1} + a}{\dfrac{x+a}{x-1} - 1}$$

$$ff = \frac{\dfrac{x+a+a(x-1)}{x-1}}{\dfrac{x+a-(x-1)}{x-1}}$$

$$ff = \frac{a+a+ax-a}{x-1} \cdot \frac{x-1}{a+a-x+1}$$

$$ff = \frac{x+ax}{x-1} \cdot \frac{x-1}{a+1}$$

$$ff = \frac{x(1+a)}{(1+a)} = x.$$

Therefore the function $f :\mapsto \dfrac{x+a}{x-1}$ is a self-inverse, since $f(f^{-1}) = x$ and the composition of the function with itself also yields x.