Practice 14

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Suggested solutions for Miscellaneous exercises 10, problems 5-20, pages 153 to 155 from Pure Mathematics 1, by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.
This is actually a selection taken from practice 14, question 5 to 14.

5.

Redefine $2x$ as ϕ; the x interval $0 \leq x \leq 180$ becomes $0 \leq 2x \leq 360$ that is $0 \leq \phi \leq 360$. Then, $3cos\phi = 2$ implies $cos\phi = \dfrac{2}{3}$; therefore, $cos^{-1}(\frac{2}{3}) = \phi = 48.20$ (using the cos^{-1} key in scientific a calculator). Now, since $cos(-\theta) = cos(\theta)$, then -48.20 is another root. Applying the periodic property of $cos(\theta \pm 360) = cos(\theta)$ we have $-48.20 + 360 = 311.80$. (Adding another $360°$ will take the angle outside the given interval ($0 \leq 2x \leq 360$). So, we have $\phi = 48.20°$ and $\phi = 311.80°$; the answer in terms of $x°$ is given by $2x° = \phi$; therefore, the two solutions in the interval are: $x° = \frac{1}{2}\phi$ or $x° = 24.1$ and $x° = 155.9$.

6.

(a) Period of the tangent and cotangent functions is $180°$; sine —$sin(bx°)$, and cosine, $cosbx°$, have period $T = \dfrac{360}{b}$; hence, when $b = 2$, $sin2x°$ and $cos2x°$ both have period equal to $180°$

(b) $sin3x° = 0.5$; interval, $0 < x < 180$; redefine $3x° = \phi$ that is, ϕ interval is three times $x°$ interval; that is, $0 < \phi < 540$. We know that the angle whose sine is 0.5 is $30°$; or, using the calculator, $sin^{-1}(0.5) = 30°$; another solution is based on the property $sin(180 - \theta) = sin(\theta)$ which implies that $180 - 30 = 150°$ is another solution. Considering that ϕ interval upper limit is $540°$, and applying the periodic property, we have two additional solutions: $30 + 360 = 390$ and $150 + 360 = 510$; then, solving for $x° = \frac{1}{3}\phi$ we have: $x_1 = 10°$; $x_2 = 50°$; $x_3 = 130°$ and $x_4 = 170°$.

7.

Given the interval: $0 \leq \theta \leq 360$ and the equation $2cos(\theta + 30)° = 1$ values of θ in the interval:
make $\phi = \theta + 30$; our interval becomes $30 \leq \phi \leq 390$. The equation $2cos(\phi) = 1$ simplifies to $cos(\phi) = 0.5$; the angle whose cosine is 0.5 is $60°$: $cos^{-1}(0.5) = 60°$. Another solution for the cosine function is given by $cos(-\theta) = cos(\theta)$; however, in our case, $-60°$ in not in the interval under consideration. Let's apply the periodic property, therefore, $-60 + 360 = 300°$. ($60 + 360 = 420°$ is outside the interval). Then, solving for $\theta = \phi - 30$ we have two solutions for the equation: $x_1 = 30°$ and $x_2 = 270°$.

8.

(a) We know that the cofunctions of complementary angles are equal to each other:
$sin(90 - \theta) = cos(\theta)$ and $cos(90 - \theta) = sin(\theta)$; that is $cos(90 - 2x) = sin(2x)$ which implies that $sin2x + cos(90 - 2x) = sin2x + sin2x = 2sin(2x)$.
(b) Hence, $2sin(2x) = -1$. Now, we define $\phi = 2x$, and the interval $0 \leq x \leq 360$ becomes $0 \leq \phi \leq 720$. Then, $sin(\phi) = -\frac{1}{2}$ and $sin^{-1}(-0.5) = -30°$; which is not in the interval; applying the periodic property of the sine function we get:$-30 + 360 = 330°$ and, considering that $sin(180 - \theta) = sin(\theta)$another solution is $180 - (-30) = 210°$; in addition to these two solutions ($300°$ and $210°$) in the interval $0°to360°$ we have to consider adding $360°$ —periodic property— since our modified interval spreads from $0°to720°$; that is: $330 + 360 = 690°$ and $210 + 360 = 570°$. Now, solving for $x°$ we have: $x° = \frac{1}{2}\phi$ which yields $x_1 = 105°$, $x_2 = 165°$, $x_3 = 285°$ and $x_4 = 345°$.

9.

We have o find the least positive value —that is, zero is not included— of the angle A for which:

(a) $sinA° = 0.2$ and $cosA° < 0$; sine is positive; cosine is negative: II quadrant. Since $sinA° = 0.2$; $sin^{-1} = 11.5°$ which is located in the I quadrant; in the second quadrant we have: $sin(180 - \theta) = sin(\theta)$; that is, $180 - 11.5 = 168.5°$.
(b) $tanA° < 0$, $sinA° < 0$: IV quadrant.
$tanA° = -0.5$; therefore, $tan^{-1}(-0.5) = -26.6$; this result is not *the least positive* value, it is indeed, negative; but $-26.6°$ is coterminal with $-26.2 + 360 = 333.4$. (c)$cosA° = sinA°$ both negative. III quadrant. Now, in the I quadrant we know that $cosA° = sinA° = \frac{\sqrt{2}}{2}$ for $A° = 45°$; in this case both, sine and cosine are negative, therefore $cosA° = sinA° = -\frac{\sqrt{2}}{2}$ and since $sin^{-1}(-\frac{\sqrt{2}}{2}) = -45°$ applying $sin(180 - \theta) = sin(\theta)$ we get $180 - (-45) = 225°$.

(d) $sinA° = -0.2275$ $A° > 360$ by calculator $sin^{-1}(-0.2275) = -13.15°$ The smallest angle whose sine is negative lies in the III quadrant, that is $sin(180 - \theta) = sin(\theta)$, then $180 - (-13.15) = 193.15$; but it is given that $A° > 360$ thus, applying the periodic property, the answer is: $193.15 + 360 = 553.15°$.

10.

(a) Proving that: $\dfrac{1}{sin\theta} - sin\theta \equiv \dfrac{cos\theta}{tan\theta}$. Combining the two terms in the left hand side:
$\dfrac{1 - sin^2\theta}{sin\theta} \equiv \dfrac{cos\theta}{tan\theta}$; from the Pythagorean identity we have: $cos^2\theta = 1 - sin^2\theta$; then,
$\dfrac{cos^2\theta}{sin\theta} \equiv \dfrac{cos\theta}{tan\theta}$; rewriting $cos^2\theta$ as $cos\theta \cdot cos\theta$ realizing that since $tan\theta = \dfrac{sin\theta}{cos\theta}$ then $\dfrac{cos\theta}{sin\theta} = \dfrac{1}{tan\theta}$ we get:
$\dfrac{cos\theta \cdot cos\theta}{sin\theta} \equiv \dfrac{cos\theta}{tan\theta}$ that is:
$cos\theta \cdot \dfrac{cos\theta}{sin\theta} \equiv \dfrac{cos\theta}{tan\theta}$ which leads to: $\dfrac{cos\theta}{tan\theta} \equiv \dfrac{cos\theta}{tan\theta}$.

(b) $\dfrac{1 - sin\theta}{cos\theta} \equiv \dfrac{cos\theta}{1 + sin\theta}$

Let's multiply the left hand side by $\dfrac{(1 + sin\theta)}{(1 + sin\theta)}$ which produces a difference of two squares $1 - sin^2\theta$ in the numerator:

$$\frac{1 - sin\theta}{cos\theta} \cdot \frac{(1 + sin\theta)}{(1 + sin\theta)} \equiv \frac{cos\theta}{1 + sin\theta}$$

$\dfrac{1 - sin^2\theta}{cos\theta \cdot (1 + sin\theta)} \equiv \dfrac{cos\theta}{1 + sin\theta}$ considering that, by the Pythagorean identity, $cos^2\theta = 1 - sin^2\theta$, we have:

$\dfrac{cos^2\theta}{cos\theta \cdot (1 + sin\theta)} \equiv \dfrac{cos\theta}{1 + sin\theta}$ which simplifies to:

$$\frac{cos\theta}{(1 + sin\theta)} \equiv \frac{cos\theta}{1 + sin\theta}$$

(c) $\dfrac{1}{tan\theta} + tan\theta \equiv \dfrac{1}{sin\theta\ cos\theta}$ combining the two terms on the left hand side:

$\dfrac{1 + tan^2\theta}{tan\theta} \equiv \dfrac{1}{sin\theta\ cos\theta}$ we know that $1 + tan^2\theta = sec^2\theta$; then,

$\dfrac{sec^2\theta}{tan\theta} \equiv \dfrac{1}{sin\theta\ cos\theta}$ substituting $\dfrac{1}{cos^2\theta} = sec^2\theta$ and $\dfrac{sin\theta}{cos\theta} = tan\theta$, the left hand side becomes:

$\dfrac{1}{cos^2\theta \cdot \dfrac{sin\theta}{cos\theta}} \equiv \dfrac{1}{sin\theta\ cos\theta}$ which simplifies to:

$$\frac{1}{sin\theta\ cos\theta} \equiv \frac{1}{sin\theta\ cos\theta}$$

(d) $\dfrac{1 - 2sin^2\theta}{cos\theta + sin\theta} \equiv cos\theta - sin\theta$ the left hand side can be rewritten as:

$\dfrac{1 - sin^2\theta - sin^2\theta}{cos\theta + sin\theta} \equiv cos\theta - sin\theta$; therefore, considering that $1 - sin^2\theta = cos^2\theta$:

$\dfrac{cos^2\theta - sin^2\theta}{cos\theta + sin\theta} \equiv cos\theta - sin\theta$; the numerator on the left hand side, a difference of two squares, becomes:

$\dfrac{(cos\theta - sin\theta)(cos\theta + sin\theta)}{cos\theta + sin\theta} \equiv cos\theta - sin\theta$; which simplifies to:

$cos\theta - sin\theta \equiv cos\theta - sin\theta$

11.

The maximum and minimum values of y and the least positive values of x at which these occur:
Note: the cosine and the sine functions $(y = cos\theta,\ \ y = sin\theta)$ reach a minimum value $y = -1$ and a maximum value $y = 1$; in summary, the range of both functions is $-1 \leq y \leq 1$.

(a) $y = 1 + cos2x$ reaches a maximum $y = 2$ whenever $cos2x = 1$; therefore, $x°$ value is given by: $cos^{-1}(1) = 0$ but we cannot consider zero as a valid solution since we are looking for *the least positive values of* x; then $0° + 360° = 360°$, thus $2x = 360°$ and $x = 180°$.
$y = 1 + cos2x$ reaches a minimum value of $y = 0$ whenever $cos2x = -1$; and it occurs at: $cos^{-1}(-1) = 180°$ therefore, $2x = 180°$ or $x = 90°$.

(b) $y = -4sin(x + 30)$; maximum, $y = 9$ occurs at $sin(x + 30) = -1$ and since $sin^{-1}(-1) = 270°$ the x values is given by: $x + 30 = 270°$ or $x = 240°$.
The minimum value of $y = 1$ occurs at $sin(x+30) = 1$; that is, $sin^{-1}(1) = 90°$ which implies that $x+30 = 90°$ or $x = 60°$.

(c) $y = 29 - 20sin(3x - 45)°$ reaches a maximum value of $y = 49$ when $sin(3x - 45) = -1$; the $x°$ value is given by: $sin^{-1}(-1) = 270°$ that is $3x - 45 = 270°$ or $x = \dfrac{270 + 45}{3} = 105°$.
Minimum value of $y = 9$ occurs when $sin(3x - 45) = 1$; in this case the $x°$ value is given by: $sin^{-1}(1) = 90°$; which implies that $x = 45°$.

(d) $y = 8 - 3cos^2x$ since $cos^2x > 0$ both, $(-1)^2 = (1)^2 = 1$; the maximum value of $y = 8$ is reached when $cosx = 0$. And, $cos^{-1}(0) = 90°$; hence, $x = 90°$.
The minimum value of $y = 5$ occurs when $cos^2x = 1$; which implies that $cosx = |1|$ or $cosx = \pm1$; since $cos^{-1}(1) = 360°$ (least positive value) and $cos^{-1}(-1) = 180°$ therefore, the least positive value of x is $180°$.

(e) $y = \dfrac{12}{3 + cosx}$ this time, the maximum value of $y = 6$ is reached when the denominator takes its smallest vale at $cosx = -1$; that is at $cos^{-1}(-1) = x = 180°$. The minimum value of $y = 3$ occurs when $cosx = 1$ therefore $cos^{-1}(1) = x = 0°$. Considering the periodic property of the cosine function the *the least positive values of* x is $0° + 360° = 360°$.

(f) $y = \dfrac{60}{1 + sin^2(2x - 15)}$ The function reaches its maximum value, $y = 60$ when $sin^2(2x-15) = 0$; that is, $sin^{-1}(0) = x = 0°$.
Solving for x we have: $2x - 15 = 0$; hence $x = \dfrac{15}{2} = 7.5°$.
The minimum value $y = 30$ occurs when $sin(2x - 15) = \pm1$; therefore, there are two possibilities: $sin^{-1}(-1) = x = -90°$. and $sin^{-1}(1) = x = 90°$. We takes the least positive vale, $90°$; then, $2x - 15 = 90$; hence, $x = \dfrac{15 + 90}{2} = 52.5°$.

12.

(a) $sin\theta = tan\theta$ The equality, $sin\theta = \dfrac{sin\theta}{cos\theta}$ is true when $cos\theta = 1$ or $sin\theta = 1$; and it occurs when $cos^{-1}(1) = x° = 0°$ or $sin^{-1}(0) = x° = 0°$ and $x° = 360°$.

(b) $2 - 2cos^2\theta = sin\theta$ considering the identity $sin^2\theta + cos^2\theta = 1$; or $cos^2\theta = 1 - sin^2\theta$, and substituting, we obtain:

$2 - 2(1 - sin^2\theta) = sin\theta$ equivalent to:

$2-2-2sin^2\theta = sin\theta$ which reduces to: $-2sin^2\theta = sin\theta$ equating it to zero in order to factorize the equation:

$$0 = sin\theta - 2sin^2\theta$$
$$0 = sin\theta(1-2sin\theta)$$

Therefore, either $sin\theta = 0$ or $1-2sin\theta = 0$; in the first instance we have as solution $sin^{-1}(0) = 0°$ also $sin\theta = 0°$ in the given interval at $x = 180°$ and $x = 360°$. The factor $1-2sin\theta$ is equal to zero when $sin\theta = \frac{1}{2}$; that is, $sin^{-1}(\frac{1}{2}) = 30°$ and, since $sin(180-\theta) = sin\theta$ another solution is $180-30 = 150°$.

(c) $tan^2\theta - 2tan\theta = 1$

This equation can be treated as a quadratic equation where the variable, say, x, is equal to $tan\theta$. By making the x substitution and setting it equal to zero we have,

$x^2-2x-1=0$ then, using the quadratic formula: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = 1$, $b = -2$ and $c = -1$ the solutions are: $x_1 = 1+\sqrt{2}$ and $x_2 = 1-\sqrt{2}$. Since we define $x = tan\theta$, thus $tan^{-1}(1-\sqrt{2}) = -22.5°$ This angle is coterminal with $337.5°$ which is a solution in the given interval $(-22.5+360 = 337.5)$. Another solution takes into consideration the periodic property of the tangent function: $tan(180+\theta) = tan\theta$, therefore $180+(-22.5) = 157.5°$. The root $tan\theta = 1+\sqrt{2}$, leads to $tan^{-1}(1+\sqrt{2}) = 67.5°$ and, again by the periodic property of the tangent function:$180+67.5 = 247.5°$.

(d) $sin2\theta - \sqrt{3}cos2\theta = 0$ since we are dealing with 2θ the given interval, $0 \leq \theta \leq 360$ becomes $0 \leq \theta \leq 720$. Then, rearranging the equation, we have:

$$sin2\theta = \sqrt{3}cos2\theta$$
$$\frac{sin2\theta}{cos2\theta} = \sqrt{3}$$
$$tan2\theta = \sqrt{3}$$
$$tan^{-1}(\sqrt{3}) = 60°$$

Two other solutions for 2θ result from the periodic property of tangent, $180+60 = 240°$ and from adding another $360°$ —we are working in the interval $0 \leq \theta \leq 720$ —: $60+360 = 420°$ and, $240+360 = 600°$. That is, we have four solutions for 2θ, namely: $60°$, $240°$, $420°$, and $600°$; therefore, the corresponding values of $\theta = \dfrac{1}{2}(2\theta)$ are: $30°$, $120°$, $210°$, and $300°$.

13.

(a) $t(x) = tan3x$ $tan\theta$ has a period of $180°$; in this case, $\theta = 3x°$; therefore, the period, $T = \dfrac{180}{3} = 60°$.

(b) Given $tan3x = \dfrac{1}{2}$ therefore, a solution is $tan^{-1}(\frac{1}{2}) = 26.6°$ another solution is given by $tan(180+\theta) = tan\theta$ that is: $26.6+180 = 206.6°$ and, the third solution results from adding $360°$: $26.6°+360° = 386.6°$; now, solving for x (we have found solutions for $3x$) we have: $x = \dfrac{1}{3}(26.6) = 8.9°$; $x = \dfrac{1}{3}(206.6) = 68.9°$; and $x = \dfrac{1}{3}(386.6) = 128.9°$.

(c) $tan3x = -\dfrac{1}{2}$; $tan^{-1}(-\frac{1}{2}) = -26.6°$ which is not a solution in the interval under consideration; but, since $tan(180+\theta) = tan\theta$ we have $180-26.6 = 153.4°$ as a solution for $3x$; the x values is $x = \dfrac{1}{3}(153.4) = 51.1°$.

(d) $tan3x = 2$; therefore, $tan^{-1}(2) = 3x = 63.4°$. The x value is given by $x = \dfrac{1}{3}(63.4) = 21.1°$.

14.

(a) We are looking for a function, sine or cosine function, of the type: $y = A + Bsinkt$. Since the period is 24 hours $k = \dfrac{360}{24} = 15$; the maximum occurs when $sin15t = 1$; the minimum, when $sin15t = -1$; then we set up a simple system of two equations:

$$3.6 = A - B \quad \text{and} \quad 6 = A + B$$

If we add these equations together, the terms containing B add up to zero; we obtain:

$$9.6 = 2A \quad \text{Hence,} \quad A = 4.8 \;\; \text{and,} \quad B = 6 - 4.8 = 1.2$$

Our equation is: $y = 4.8 + 1.2sinkt$
Note: similarly, we can model the phenomena using the cosine function. Or, instead of $y = A + Bsinkt$ we may use $y = A - Bsinkt$, considering that the maximum value is reached when $sinkt = -1$ and the minimum value when $sinkt = 1$, etc.

(b) Using the same approach as in (a): $k = \frac{360}{10} = 36$; using he equations of the form $y = A + Bsin36t$ our system of two equation is,

$$15000 = A - B \;\; \text{and} \quad 28000 = A + B$$

$$43000 = 2A \quad \text{That is,} \quad A = 21500 \;\; \text{and,} \quad B = 28000 - 21500 = 6500$$

$$y = 21500 + 6500sin36t$$

(c) $k = \frac{360}{360} = 1$; $y = A + Bsint$ The system of two equations:

$$2 = A - B \;\; \text{and} \quad 22 = A + B$$

$$24 = 2A \quad \text{therefore} \quad A = 12, \quad B = 10$$

Our equation becomes,

$$y = 12 + 10sint$$

15.