Practice 12

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Suggested solutions for Exercises 9B (number 6 to 13, p 135) and Miscellaneous exercise 9 from number 1 to 33, pages 135 to 137, from Pure Mathematics 1 by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

The expansion of $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$; while, $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$ or in a scientific calculator use the nCr key —which is equivalent to n choose r or $\binom{n}{r}$.

Notice that: (i) the powers of a and b add up to n; (ii) there are $n+1$ terms in the expansion of $(a+b)^n$; (iii) $j+1$ is equal to the term number; that is, the third term has coefficient $\binom{n}{2}$, the 7th term, coefficient $\binom{n}{6}$, etc; (iv) $1^n = 1$, and $x^0 = 1$; still, for the sake of clarity, those terms are being included in the expansions.

6.

(a) $(1+x)^{22} = \binom{22}{0}1^{22}x^0 + \binom{22}{1}1^{21}x^1 + \binom{22}{2}1^{20}x^2 + ...$ which leads to: $(1+x)^{22} = 1 + 22x + 231x^2 + ...$

(b) $(1-x)^{30} = \binom{30}{0}1^{30}(-x)^0 + \binom{30}{1}1^{29}(-x)^1 + \binom{30}{2}1^{28}(-x)^2 + ...$ or: $(1-x)^{30} = 1 - 30x + 435x^2 + ...$

(c) $(1-4x)^{18} = \binom{18}{0}1^{18}(-4x)^0 + \binom{18}{1}1^{17}(-4x)^1 + \binom{18}{2}1^{16}(-4x)^2 + ...$ or: $(1-4x)^{18} = 1 - 72x + 2448x^2 + ...$

(d)$(1+6x)^{19} = \binom{19}{0}1^{19}(6x)^0 + \binom{19}{1}1^{18}(6x)^1 + \binom{19}{2}1^{17}(6x)^2 + ...$ or: $(1+6x)^{19} = 1 + 114x + 6156x^2 + ...$

7.

$(1+2x)^8 = \binom{8}{0}1^8(2x)^0 + \binom{8}{1}1^7(2x)^1 + \binom{8}{2}1^6(2x)^2 + ... = 1 + 16x + 112x^2$.
Setting $x = 0.001$ then $(1+2(0.01))^8 = (1.02)^8 = 1 + 16(0.01) + 112(0.001) = 1.17$.

Notice that an x value much smaller than one makes high powers of x negligible; for instance, $(0.01)^3 = 0.000001$.

8.

$(2+5x)^{12} = \binom{12}{0}2^{12}(5x)^0 + \binom{12}{1}2^{11}(5x)^1 + \binom{12}{2}2^{10}(5x)^2 = 4096 + 122880x + 1689600x^2$.

In order to find the x value to approximate 2.005^{12} the quantity inside the parenthesis has to be equal to 2.005; that is, $2 + 5x = 2.005$. Solving for x we have: $x = \dfrac{2.005-2}{5} = 0.001$; substituting in $x = 0.001$ into the expansion of $(2+5x)^{12}$, we get: $2.005^{12} = 4096 + 122880(0.001) + 1689600(0.001)^2 = 4220.57$.

9.

$(1+2x)^{16} = \binom{16}{0}1^{16}(2x)^0 + \binom{16}{1}1^{15}(2x)^1 + \binom{16}{2}1^14(2x)^2 + \binom{16}{3}1^{13}(2x)^3 + ... = 1 + 32x + 480x^2 + 4480x^3 + ...$

According to the distributive property of multiplication over addition, $(a+b)(c+d) = a(c+d) + b(c+d)$, in our case $(1+3x)(1+32x+480x^2+4480x^3)$ becomes $1(1+32x+480x^2+4480x^3) + 3x(1+32x+480x^2+4480x^3)$. In the first product the term including x^3 is $4480x^3$, in the second product, $3x \cdot 480x^2 = 1440x^3$; therefore, the coefficient of x^3 is given by $4480 + 1440 = 5920$.

10.

$(1-3x)^{10} = \binom{10}{0}1^{10}(-3x)^0 + \binom{10}{1}1^9(-3x)^1 + \binom{10}{2}1^8(-3x)^2 + ... = 1 - 30x + 405x^2$.

Also, $(1+3x)^2 = 1 + 6x + 9x^2$; then, by the distributive property of multiplication over addition:

$$(1+6x+9x^2)(1-30x+405x^2) = 1(1-30x+405x^2) + 6x(1-30x+405x^2) + 9x^2(1-30x+405x^2)$$

Each of the three products generate a term including x^2; they are $1 \cdot 405x^2 = 405x^2$, $6x \cdot (-30x) = -180x^2$ and $9x^2 \cdot 1 = 9x^2$; therefore, the coefficient of x^2 is given by $405 - 180 + 9 = 234$.

11.

In order to find the value of a that makes the coefficient of x equal to 207, we need to expand $(1+5x)^{40}$ to the term that includes x, which is the second term: $(1+5x)^{40} = \binom{40}{0}1^{40}(5x)^0 + \binom{40}{1}1^{39}(5x)^1 = 1 + 200x$. Then,

$$(1+ax)(1+200x) = 1(1+200x) + ax(1+200x) = 1 + 200x + ax + 200ax^2$$

The coefficient of x s given by $200x + ax = 207$ or $200 + a = 207$ that is, $a = 7$.

12.

The expansions of $(1-x)^8$ and $(1+x)^8$ differ in the odd powers, since $(-x)^{odd}$ leads to a negative coefficient; while $(+x)^{odd}$ yields a positive number; therefore, the sum $(1-x)^8 + (1+x)^8$ cancels out the odd powers terms, and duplicate the even powers terms since $(-x)^{even} = (+x)^{even}$; therefore,

$$(1-x)^8 + (1+x)^8 = 2\left[\binom{8}{0}1^8 + \binom{8}{2}1^6x^2 + \binom{8}{4}1^2x^4 + \binom{8}{6}1^2x^6 + \binom{8}{8}x^8\right]$$

$$= 2 + 56x^2 + 140x^4 + 56x^6 + 2x^8$$

In order for $(1-x)^8 + (1+x)^8$ to be equal to $0.99^8 + 1.01^8$ then $1 - x = 0.99$ and $1 + x = 1.01$, that is, the suitable value of x is 0.01. Then, by substituting $x = 0.01$ into the previous result we get:

$$0.99^8 + 1.01^8 = 2 + 56(0.01)^2 + 140(0.01)^4 + 56(0.01)^6 + 2(0.01)^8 = 2.0056014000560002.$$

13.

We are given that $(1+ax)^n = 1 + 36x + 576x^2 + ...$ Then,

$$\binom{n}{0}1^n(ax)^0 + \binom{n}{1}1^{(n-1)}(ax)^1 + \binom{n}{2}1^{(n-2)}(ax)^2 = 1 + nax + \frac{n!}{2!(n-2)!}(a^2x^2)$$

$$1 + nax + \frac{n(n-1)(n-2)!}{2!(n-2)!}(a^2x^2) = 1 + nax + \frac{n(n-1)}{2}a^2x^2$$

The previous result has to be equal to the given expansion:

$$1 + nax + \frac{n(n-1)}{2}a^2x^2 = 1 + 36x + 576x^2$$

That is, coefficients on the left side of the equation have to be equal to coefficients on the right side of the equation:

$$na = 36 \quad \text{and} \quad \frac{n(n-1)a^2}{2} = 576$$

Solving for a in $na = 36$ we have $a = \frac{36}{n}$, then:

$$\frac{n(n-1)}{2}\left(\frac{36}{n}\right)^2 = 576$$
$$\frac{n(n-1)1296}{2n^2} = 576$$
$$\frac{(n-1)1296}{2n} = 576$$
$$(n-1)1296 = 1152n$$
$$1296n - 1152n = 1296$$
$$144n = 1296 \quad \text{or} \quad n = 9$$

Then, since $a = \frac{36}{n} = \frac{36}{9} = 4$.

Miscellaneous exercise 9

1.

$$(3+4x)^3 = \binom{3}{0}3^3(4x)^0 + \binom{3}{1}3^2(4x)^1 + \binom{3}{2}3^1(4x)^2 + \binom{3}{0}3^0(4x)^3 = 27 + 108x + 144x^2 + 64x^3$$

2.

(a)

$$(1+4x)^{10} = \binom{10}{0}1^{10}(4x)^0 + \binom{10}{1}1^9(4x)^1 + \binom{10}{2}1^8(4x)^2 + ...$$
$$(1+4x)^{10} = 1 + 40x + 120x^2 + ...$$

(b)

$$(1-2x)^{16} = \binom{16}{0}1^{16}(-2x)^0 + \binom{16}{1}1^{15}(-2x)^1 + \binom{16}{2}1^{14}(-2x)^2 + ...$$
$$(1-2x)^{16} = 1 - 32x + 480x^2 + ...$$

3.

(a) Coefficient of a^3b^5: the power of b in the expansion of $(a+b)^n$ is given by j. In our case, $(3a-2b)^8 = \sum_{j=0}^{8}\binom{8}{j}(3a)^{n-j}(-2b)^j$. Therefore the sixth term is given by: $\binom{8}{5}(3a)^3(-2b)^5 = 56 \cdot 3^3a^3 \cdot (-2)^5b^5 = -48348a^3b^5$.

(b) Again, $j = 5$; then,

$$\binom{8}{5}(5a)^3(\frac{1}{2}b)^5 = 56 \cdot 125a^3 \cdot \frac{1}{32}b^5 = \frac{875}{4}a^3b^5.$$

4.

$(3+5x)^7 = \binom{7}{0}3^7(5x)^0 + \binom{7}{1}3^6(5x)^1 + \binom{7}{2}3^5(5x)^2 + ... = 2187 + 25515x + 127575x^2$

By setting $x = 0.01$, then:

$[3+5(0.01)]^7 = (3.05)^7 \approx 2187 + 25515(0.01) + 127575(0.001)^2 = 2454.9 \approx 2455.$

5.

$(2+\frac{1}{4}x)^8 = \binom{8}{0}2^8(\frac{1}{4}x)^0 + \binom{8}{1}2^7(\frac{1}{4}x)^1 + \binom{8}{2}2^6(\frac{1}{4}x)^2 + \binom{8}{3}2^5(\frac{1}{4}x)^3 + ...$

$(2+\frac{1}{4}x)^8 = 256 + 256x + 112x^2 + 28x^3 + ...$

In order to approximate 2.0025^8 by using $(2+\frac{1}{4}x)^8$, then:

$$2 + \frac{1}{4}x = 2.0025$$
$$x = 4(2.0025-2) = 0.01$$

$$\left[2+\frac{1}{4}(0.01)\right]^8 \approx 256 + 256(0.01) + 112(0.01)^2 + 28(0.01)^3$$
$$(2.0025)^8 \approx 256 + 2.56 + 0.0112 + 0.000028 \approx 258.571$$

6.

$(2-3x)^8 = \binom{8}{0}2^8(-3x)^0 + \binom{8}{1}2^7(-3x)^1 + \binom{8}{2}2^6(-3x)^2$

$(2-3x)^8 = 256 - 3072x + 16128x^2$
Since we have to set $((2-3x)^8 = (1.997)^8$, then:

$2-3x = 1.997$ which implies that $x = \dfrac{2-1.997}{3} = 0.001.$

$$[2-3(0.001)]^8 \approx 256 - 3072(0.001) + 16128(0.001)^2$$
$$(1.997)^8 \approx 252.94 \approx 253.$$

7.

$(x^2+\frac{1}{x})^3 = \binom{3}{0}(x^2)^3(\frac{1}{x})^0 + \binom{3}{1}(x^2)^2(\frac{1}{x}) + \binom{3}{2}x^2(\frac{1}{x})^2 + \binom{3}{3}(x^2)^0(\frac{1}{x})^3.$

$(x^2+\frac{1}{x})^3 = x^6 + 3(x^4)(\frac{1}{x} + 3x^2\frac{1}{x^2} + \frac{1}{x^3}$

$(x^2+\frac{1}{x})^3 = x^6 + 3x^3 + 3 + \frac{1}{x^3}$

8.

$$\left(2x - \frac{3}{x^2}\right)^4 = \binom{4}{0}(2x)^4(-\tfrac{3}{x^2})^0 + \binom{4}{1}(2x)^3(-\tfrac{3}{x^2})^1 + \binom{4}{2}(2x)^2(-\tfrac{3}{x^2})^2 + \binom{4}{3}(2x)^1(-\tfrac{3}{x^2})^3 + \binom{4}{4}(2x)^0(-\tfrac{3}{x^2})^4$$
$$\left(2x - \frac{3}{x^2}\right)^4 = 16x^4 + 4(8x^3)(-\tfrac{3}{x^2}) + 6(4x^2)(-\tfrac{9}{x^4}) + 4(2x)(-\tfrac{27}{x^6}) + \tfrac{81}{x^8}$$
$$\left(2x - \frac{3}{x^2}\right)^4 = 16x^4 - 96x + \tfrac{216}{x^2} - \tfrac{216}{x^5} + \tfrac{81}{x^8}.$$

9.

The expansions of $\left(x - \frac{1}{2x}\right)^6$ and $\left(x + \frac{1}{2x}\right)^6$ differ by the 3rd and 5th terms, since $-\frac{1}{2x}$ raised to an odd power yield a negative coefficient; therefore, those terms cancel each other out; all other terms are duplicated, as follow:

$$\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2\tbinom{6}{0}x^6 + 2\tbinom{6}{2}x^4(\frac{1}{2x})^2 + 2\binom{6}{4}x^2(\frac{1}{2x})^4 + 2\binom{6}{6}x^0(\frac{1}{2x})^6$$
$$\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2x^6 + 2(15)x^4 \cdot (\frac{1}{4x^2}) + 2(15)x^2 \cdot (\frac{1}{16x^4}) + 2(1) \cdot (\frac{1}{64x^6})$$
$$\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2x^6 + \frac{15}{2}x^2 + \frac{15}{8}\frac{1}{x^2} + \frac{1}{32}x^6.$$

10.

The coefficient of x^2 in the expansion of $(x^4 + \frac{4}{x})^3$ is given by the term $\binom{3}{j}(x^4)^{3-j}(\frac{4}{x})^j$, that is,

$$(x^4)^{3-j}(\frac{4}{x})^j = x^2$$
$$x^{4(3-j)} \cdot x^{(-1)(j)} = x^2$$

Therefore, the exponents on the left side of the equation add up to 2:

$$12 - 4j + (-j) = 2$$
$$12 - 5j = 2$$
$$j = 2$$

The coefficient is $\binom{3}{2}(x^4)^{3-2}(\frac{4}{x})^2 = 3 \cdot x^4 \cdot \frac{16}{x^2} = 48x^2$.

11.

The term independent of x occurs when the powers of $2x$ and $(\frac{5}{x})$ are equal to one another. That implies that $(n - j) = j$ following the binomial expansion $(2x + \frac{5}{x})^6 = \sum_{j=0}^{6} \binom{6}{j}(2x)^{6-j}(\frac{5}{x})^j$; therefore, since $(n - j) = j$ then $n = 6 = 2j$ or $j = 3$. The term independent of x is given by:

$$\binom{6}{3}(2x)^3(\frac{5}{x})^3 = 20 \cdot 8x^3 \cdot \frac{125}{x^3} = 20000.$$