Suggested solutions for Exercises 9B (number 6 to 13, p 135) and Miscellaneous exercise 9 from number 1 to 33, pages 135 to 137, from Pure Mathematics 1 by Hugh Neil and Douglas Qualing, Cambridge University Press, 2002.

The expansion of \((a + b)^n\) is given by \(\sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j\); while, \(\binom{n}{j} = \frac{n!}{j!(n-j)!}\) or in a scientific calculator use the \(nCr\) key — which is equivalent to \(n\) choose \(r\) or \(\binom{n}{r}\).

Notice that: (i) the powers of \(a\) and \(b\) add up to \(n\); (ii) there are \(n + 1\) terms in the expansion of \((a + b)^n\); (iii) \(j + 1\) is equal to the term number; that is, the third term has coefficient \(\binom{2}{2}\), the 7th term, coefficient \(\binom{6}{2}\), etc; (iv) \(1^n = 1\), and \(x^0 = 1\); still, for the sake of clarity, those terms are being included in the expansions.

6.

(a) \((1 + x)^{22} = \binom{22}{0}1^{22}x^0 + \binom{22}{1}1^{21}x^1 + \binom{22}{2}1^{20}x^2 + \ldots\) which leads to: \((1 + x)^{22} = 1 + 22x + 231x^2 + \ldots\)

(b) \((1 - x)^{30} = \binom{30}{0}1^{30}(-x)^0 + \binom{30}{1}1^{29}(-x)^1 + \binom{30}{2}1^{28}(-x)^2 + \ldots\) or: \((1 - x)^{30} = 1 - 30x + 435x^2 + \ldots\)

(c) \((1 - 4x)^{18} = \binom{18}{0}1^{18}(-4x)^0 + \binom{18}{1}1^{17}(-4x)^1 + \binom{18}{2}1^{16}(-4x)^2 + \ldots\) or: \((1 - 4x)^{18} = 1 - 72x + 2448x^2 + \ldots\)

(d) \((1 + 6x)^{19} = \binom{19}{0}1^{19}(6x)^0 + \binom{19}{1}1^{18}(6x)^1 + \binom{19}{2}1^{17}(6x)^2 + \ldots\) or: \((1 + 6x)^{19} = 1 + 114x + 6156x^2 + \ldots\)

7.

\((1 + 2x)^8 = \binom{8}{0}1^8(2x)^0 + \binom{8}{1}1^7(2x)^1 + \binom{8}{2}1^6(2x)^2 + \ldots = 1 + 16x + 112x^2\)

Setting \(x = 0.001\) then \((1 + 2(0.01))^8 = (1.02)^8 = 1 + 16(0.01) + 112(0.001) = 1.17\).

Notice that an \(x\) value much smaller than one makes high powers of \(x\) negligible; for instance, \((0.01)^3 = 0.000001\).

8.

\((2 + 5x)^{12} = \binom{12}{0}2^{12}(5x)^0 + \binom{12}{1}2^{11}(5x)^1 + \binom{12}{2}2^{10}(5x)^2 = 4096 + 122880x + 1689600x^2\).

In order to find the \(x\) value to approximate 2.005\(^{12}\) the quantity inside the parenthesis has to be equal to 2.005; that is, \(2 + 5x = 2.005\). Solving for \(x\) we have: \(x = \frac{2.005 - 2}{5} = 0.001\); substituting in \(x = 0.001\) into the expansion of \((2 + 5x)^{12}\), we get: \(2.005^{12} = 4096 + 122880(0.001) + 1689600(0.001)^2 = 4220.57\).

9.

\((1 + 2x)^{16} = \binom{16}{0}1^{16}(2x)^0 + \binom{16}{1}1^{15}(2x)^1 + \binom{16}{2}1^{14}(2x)^2 + \binom{16}{3}1^{13}(2x)^3 + \ldots = 1 + 32x + 480x^2 + 4480x^3 + \ldots\)

According to the distributive property of multiplication over addition, \((a + b)(c + d) = a(c + d) + b(c + d)\), in our case \((1 + 3x)(1 + 32x + 480x^2 + 4480x^3)\) becomes \(1(1 + 32x + 480x^2 + 4480x^3) + 3x(1 + 32x + 480x^2 + 4480x^3)\). In the first product the term including \(x^3\) is 4480\(x^3\), in the second product, \(3x \cdot 480x^2 = 1440x^3\); therefore, the coefficient of \(x^3\) is given by 4480 + 1440 = 5920.
(1 - 3x)^{10} = \binom{10}{0}1^{10}(-3x)^0 + \binom{10}{1}1^{9}(-3x)^1 + \binom{10}{2}1^{8}(-3x)^2 + \ldots = 1 - 30x + 405x^2.

Also, (1 + 3x)^2 = 1 + 6x + 9x^2; then, by the distributive property of multiplication over addition:

\[(1 + 6x + 9x^2)(1 - 30x + 405x^2) = 1(1 - 30x + 405x^2) + 6x(1 - 30x + 405x^2) + 9x^2(1 - 30x + 405x^2)\]

Each of the three products generate a term including \(x^2\); they are \(1 \cdot 405x^2 = 405x^2\), \(6x \cdot (-30x) = -180x^2\) and \(9x^2 \cdot 1 = 9x^2\); therefore, the coefficient of \(x^2\) is given by \(405 - 180 + 9 = 234\).

11.

In order to find the value of \(a\) that makes the coefficient of \(x\) equal to 207, we need to expand \((1 + 5x)^{40}\) to the term that includes \(x\), which is the second term: \((1 + 5x)^{40} = \left(\frac{1}{1}\right)^{10}(5x)^0 + \binom{40}{1}1^{39}(5x)^1 = 1 + 200x\). Then,

\[(1 + ax)(1 + 200x) = 1(1 + 200x) + ax(1 + 200x) = 1 + 200x + ax + 200ax^2\]

The coefficient of \(x\) is given by \(200 + a = 207\) or \(200 + a = 207\) that is, \(a = 7\).

12.

The expansions of \((1 - x)^8\) and \((1 + x)^8\) differ in the odd powers, since \((-x)^{odd}\) leads to a negative coefficient; while \((+x)^{odd}\) yields a positive number; therefore, the sum \((1 - x)^8 + (1 + x)^8\) cancels out the odd powers terms, and duplicate the even powers terms since \((-x)^{even} = (+x)^{even}\); therefore,

\[\begin{align*}
(1 - x)^8 + (1 + x)^8 &= 2 \left[ \binom{8}{0}1^8 + \binom{8}{2}1^6x^2 + \binom{8}{4}1^4x^4 + \binom{8}{6}1^2x^6 + \binom{8}{8}x^8 \right] \\
&= 2 + 56x^2 + 140x^4 + 56x^6 + 2x^8
\end{align*}\]

In order for \((1 - x)^8 + (1 + x)^8\) to be equal to 0.99^8 + 1.01^8 then \(1 - x = 0.99\) and \(1 + x = 1.01\), that is, the suitable value of \(x\) is 0.01. Then, by substituting \(x = 0.01\) into the previous result we get:

\[0.99^8 + 1.01^8 = 2 + 56(0.01)^2 + 140(0.01)^4 + 56(0.01)^6 + 2(0.01)^8 = 2.0056014000560002.\]

13.

We are given that \((1 + ax)^n = 1 + 36x + 576x^2 + \ldots\). Then,

\[\begin{align*}
\binom{n}{0}1^n(ax)^0 + \binom{n}{1}1^{n-1}(ax)^1 + \binom{n}{2}1^{n-2}(ax)^2 &= 1 + nax + \frac{n!}{2!(n-2)!}(a^2x^2) \\
1 + nax + \frac{n(n-1)(n-2)!}{2!(n-2)!}(a^2x^2) &= 1 + nax + \frac{n(n-1)}{2}a^2x^2
\end{align*}\]

The previous result has to be equal to the given expansion:

\[1 + nax + \frac{n(n-1)}{2}a^2x^2 = 1 + 36x + 576x^2\]

That is, coefficients on the left side of the equation have to be equal to coefficients on the right side of the equation:

\[na = 36\text{ and }\frac{n(n-1)a^2}{2} = 576\]
Solving for \( a \) in \( na = 36 \) we have \( a = \frac{36}{n} \), then:

\[
\frac{n(n - 1)}{2} \left( \frac{36}{n} \right)^2 = 576
\]
\[
n(n - 1) \frac{1296}{2n^2} = 576
\]
\[
\frac{(n - 1)1296}{2n} = 576
\]
\[
(n - 1)1296 = 1152n
\]
\[
1296n - 1152n = 1296
\]
\[
144n = 1296 \quad \text{or} \quad n = 9
\]

Then, since \( a = \frac{36}{n} = \frac{36}{9} = 4 \).

**Miscellaneous exercise 9**

1.

\[
(3 + 4x)^3 = \binom{3}{0}3^0(4x)^0 + \binom{3}{1}3^1(4x)^1 + \binom{3}{2}3^2(4x)^2 + \binom{3}{0}3^0(4x)^3 = 27 + 108x + 144x^2 + 64x^3
\]

2.

(a)

\[
(1 + 4x)^10 = \binom{10}{0}1^{10}(4x)^0 + \binom{10}{1}1^{9}(4x)^1 + \binom{10}{2}1^8(4x)^2 + ... \\
(1 + 4x)^10 = 1 + 40x + 120x^2 + ...
\]

(b)

\[
(1 - 2x)^16 = \binom{16}{0}1^{16}(-2x)^0 + \binom{16}{1}1^{15}(-2x)^1 + \binom{16}{2}1^{14}(-2x)^2 + ... \\
(1 - 2x)^16 = 1 - 32x + 480x^2 + ...
\]

3.

(a) Coefficient of \( a^3b^5 \): the power of \( b \) in the expansion of \((a + b)^n\) is given by \( j \). In our case, \((3a - 2b)^8 = \sum_{j=0}^{8} \binom{8}{j}(3a)^{n-j}(-2b)^j\). Therefore the sixth term is given by: \( \binom{8}{5}(3a)^3(-2b)^5 = 56 \cdot 3^3a^3 \cdot (-2)^5b^5 = -48348a^3b^5 \).

(b) Again, \( j = 5 \); then,

\[
\binom{8}{5}(5a)^3\left(\frac{1}{2}b\right)^5 = 56 \cdot 125a^3 \cdot \frac{1}{32}b^5 = \frac{875}{4}a^3b^5.
\]
4. 

\[(3 + 5x)^7 = (\binom{7}{0})3^7(5x)^0 + (\binom{7}{1})3^6(5x)^1 + (\binom{7}{2})3^5(5x)^2 + \ldots = 2187 + 25515x + 127575x^2\]

By setting \(x = 0.01\), then:

\[\left[3 + 5(0.01)\right]^7 = (3.05)^7 \approx 2187 + 25515(0.01) + 127575(0.001)^2 = 2454.9 \approx 2455.\]

5. 

\[(2 + \frac{1}{4}x)^8 = (\binom{8}{0})2^8(\frac{1}{4}x)^0 + (\binom{8}{1})2^7(\frac{1}{4}x)^1 + (\binom{8}{2})2^6(\frac{1}{4}x)^2 + (\binom{8}{3})2^5(\frac{1}{4}x)^3 + \ldots\]

\[2 + \frac{1}{4}x = 2.0025\]

\[x = 4(2.0025 - 2) = 0.01\]

\[\left[2 + \frac{1}{4}(0.01)\right]^8 \approx 256 + 256(0.01) + 112(0.01)^2 + 28(0.01)^3\]

\[(2.0025)^8 \approx 256 + 2.56 + 0.0112 + 0.000028 \approx 258.571\]

6. 

\[(2 - 3x)^8 = (\binom{8}{0})2^8(-3x)^0 + (\binom{8}{1})2^7(-3x)^1 + (\binom{8}{2})2^6(-3x)^2\]

\[(2 - 3x)^8 = 256 - 3072x + 16128x^2\]

Since we have to set \((2 - 3x)^8 = (1.997)^8\), then:

\[2 - 3x = 1.997 \quad \text{which implies that} \quad x = \frac{2 - 1.997}{3} = 0.001.\]

\[\left[2 - 3(0.001)\right]^8 = 256 - 3072(0.001) + 16128(0.001)^2\]

\[(1.997)^8 \approx 252.94 \approx 253.\]

7. 

\[(x^2 + \frac{1}{x})^3 = (\binom{3}{0})(x^2)^3(\frac{1}{x})^0 + (\binom{3}{1})(x^2)^2(\frac{1}{x})^1 + (\binom{3}{2})(x^2)(\frac{1}{x})^2 + (\binom{3}{3})(x^2)^0(\frac{1}{x})^3\]

\[(x^2 + \frac{1}{x})^3 = x^6 + 3(x^4)(\frac{1}{x}) + 3(x^2)(\frac{1}{x})^2 + \frac{1}{x^3}\]

\[(x^2 + \frac{1}{x})^3 = x^6 + 3x^3 + 3 + \frac{1}{x^3}\]
8.

\[
\left(2x - \frac{3}{x^2}\right)^4 = \binom{4}{0}(2x)^4(-\frac{3}{x^2})^0 + \binom{4}{1}(2x)^3(-\frac{3}{x^2})^1 + \binom{4}{2}(2x)^2(-\frac{3}{x^2})^2 + \binom{4}{3}(2x)^1(-\frac{3}{x^2})^3 + \binom{4}{4}(2x)^0(-\frac{3}{x^2})^4
\]

\[
\left(2x - \frac{3}{x^2}\right)^4 = 16x^4 + 4(8x^3)(-\frac{3}{x^2}) + 6(4x^2)(-\frac{9}{x^3}) + 4(2x)(-\frac{27}{x^4}) + \frac{81}{x^6}
\]

\[
\left(2x - \frac{3}{x^2}\right)^4 = 16x^4 - 96x + \frac{216}{x^2} - \frac{216}{x^4} + \frac{81}{x^6}.
\]

9.

The expansions of \(\left(x - \frac{1}{2x}\right)^6\) and \((x + \frac{1}{2x})^6\) differ by the 3rd and 5th terms, since \(-\frac{1}{2x}\) raised to an odd power yield a negative coefficient; therefore, those terms cancel each other out; all other terms are duplicated, as follow:

\[
\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2\binom{6}{0}x^6 + 2\binom{6}{2}x^4(\frac{1}{2x})^2 + 2\binom{6}{4}x^2(\frac{1}{2x})^4 + 2\binom{6}{6}x^0(\frac{1}{2x})^6
\]

\[
\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2x^6 + 2(15)x^4 \cdot \left(\frac{1}{4x^2}\right) + 2(15)x^2 \cdot \left(\frac{1}{16x^4}\right) + 2(1) \cdot \left(\frac{1}{64x^6}\right)
\]

\[
\left(x - \frac{1}{2x}\right)^6 + \left(x + \frac{1}{2x}\right)^6 = 2x^6 + \frac{15}{2}x^4 + \frac{15}{8}x^2 + \frac{1}{32}x^6.
\]

10.

The coefficient of \(x^2\) in the expansion of \((x^4 + \frac{1}{x})^3\) is given by the term \(\binom{3}{j}(x^4)^{3-j}(\frac{1}{x})^j\), that is,

\[
(x^4)^{3-j}(\frac{1}{x})^j = x^2
\]

\[
x^{4(3-j)} \cdot x^{-1(j)} = x^2
\]

Therefore, the exponents on the left side of the equation add up to 2:

\[
12 - 4j + (-j) = 2
\]

\[
12 - 5j = 2
\]

\[
j = 2
\]

The coefficient is \(\binom{3}{2}(x^4)^{3-2}(\frac{1}{x})^2 = 3 \cdot x^4 \cdot \frac{16}{x^2} = 48x^2\).

11.

The term independent of \(x\) occurs when the powers of \(2x\) and \((\frac{5}{x})\) are equal to one another. That implies that \((n - j) = j\) following the binomial expansion \((2x + \frac{5}{x})^6 = \sum_{j=0}^{6} \binom{6}{j}(2x)^{6-j}(\frac{5}{x})^j\); therefore, since \((n - j) = j\) then \(n = 6 = 2j\) or \(j = 3\). The term independent of \(x\) is given by:

\[
\binom{6}{3}(2x)^3(\frac{5}{x})^3 = 20 \cdot 8x^3 \cdot \frac{125}{x^3} = 20000.
\]