Practice 11

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Suggested solutions for problems 8C, 8 to 11, page 125; and Miscellaneous exercises 8, problems 1- 15, pages 125 and 126 from Pure Mathematics 1, by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

8.

(a) $l = a_1 + (n-1)d$ substituting in the formula a_1 and d values: $l = 5 + 19(3) = 62.$
(b)

$$S = \frac{n}{2}[2a_0 + (n-1)d] > 1000$$

Multiplying the equation through out by 2, and substituting in the values, we have:

$$n[2(5) + (n-1)3] > 2000$$
$$(10 + 3n - 3) > 2000$$
$$3n^2 + 7n > 2000$$
$$3n^2 + 7n - 2000 > 0$$

By using the quadratic formula or otherwise, we obtain the two values of n that satisfies the equation: $n_1 = 24.68$ and $n_2 = -27.01$. Disregard the negative result, so since we are looking for the number of days, at $n = 24.68$ the equation is actually equal to zero, greater than zero for the integer value $n = 25$.

9.

(a) $l = a_1 + (n-1)d$ substituting in the formula a_1 and d values: $l = 30 + 23(2) = 76.$ (b)

$$S = \frac{n}{2}[a_0 + l]$$

$$S = \frac{24}{2}[30 + 76] = 1272.$$

10.

(a)

$$S = \frac{n}{2}(a_0 + l) \quad = \frac{100}{2}(1 + 100) = 5050.$$

(b)

$$S = \frac{n}{2}(a_0 + l) \quad = \frac{100}{2}(101 + 200) = 15050.$$

(c) How many terms from $n+1$ to $2n$? Subtracting the larger minus the smaller and adding 1 (the extremes, n, and $n+1$ are included) we get: $2n - (n+1) = 2n - n - 1 = n - 1$ and $n - 1 + 1 = n$, then:

$$S_n = \frac{n}{2}[(n+1) + 2n]$$
$$Sn = \frac{n}{2}(3n + 1)$$

11.

$$l = a_1 + (n-1)d \quad = 300 + 15(80) = 42000$$

From 2000 to 2015, there are $2015 - 2000 + 1$ years, that is $n = 16$:

$$S_{2015} = \frac{n}{2}(a_0 + l) = \frac{16}{2}(30000 + 42000) = 576000$$

From 2016 to 2040, $n = 25$, the employee remains on the same salary (\$42000.00):

$$25(42000) = \$1050000$$

The grand total is:

$$= 1050000 + 576000 = 1626000.00$$

Miscellaneous exercise 8

1.

The sequence is defined inductively by $u_{r+1} = 3u_r - 1$, $u_0 = c$:
(a)
(i) $c = 1$ $u_0 = 1$ then: $u_1 = 3u_0 - 1 = 3(1) - 1 = 2$; similarly, $u_2 = 3u_1 - 1 = 3(2) - 1 = 5$ and $u_3 = 3(5) - 1 = 14$; $u_4 = 3(14) - 1 = 41$.
(ii) $c = 2$ $u_0 = 2$ $u_1 = 3u_0 - 1 = 3(2) - 1 = 5$; $u_2 = 3(5) - 1 = 14$; $u_3 = 3(14) - 1 = 41$; $u_4 = 3(41) - 1 = 122$.
(iii) $c = 0$ $u_0 = 0$; $u_1 = 3u_0 - 1 = 3(0) - 1 = -1$; $u_2 = 3(-1) - 1 = -4$;$u_3 = 3(-4) - 1 = -13$;$u_4 = 3(-13) - 1 = -40$;
(iv) $c = \frac{1}{2}$ $u_0 = \frac{1}{2}$, $u_1 = 3u_0 - 1 = 3(\frac{1}{2}) - 1 = \frac{1}{2}$; so the terms of the sequence for $c = \frac{1}{2}$ are all equal to $\frac{1}{2}$.

(b)
(i) $u_0 = 1$, $u_r = \frac{1}{2} + b \cdot 3^r$ $1 = \frac{1}{2} + b \cdot 3^0$ therefore, $b = \frac{1}{2}$.
(ii) $u_0 = 2$, $u_r = \frac{1}{2} + b \cdot 3^r$ $2 = \frac{1}{2} + b \cdot 3^0$ therefore, $b = \frac{3}{2}$.
(iii) $u_0 = 0$, $u_r = \frac{1}{2} + b \cdot 3^r$ $0 = \frac{1}{2} + b \cdot 3^0$ therefore, $b = -\frac{1}{2}$.
(iv) $u_0 = \frac{1}{2}$, $u_r = \frac{1}{2} + b \cdot 3^r$ $\frac{1}{2} = \frac{1}{2} + b \cdot 3^0$ therefore, $b = 0$.

(c)

$$u_r = \frac{1}{2} + b \cdot 3^r$$
$$u_{r+1} = 3u_r - 1$$

Substituting the first equation into the second, it yields:

$$u_{r+1} = 3[\frac{1}{2} + b \cdot 3^r] - 1$$
$$u_{r+1} = \frac{3}{2} + b \cdot 3^1 \cdot 3^r - 1$$
$$u_{r+1} = \frac{1}{2} + b \cdot 3^{r+1}$$

2.

$u_1 = 0$, $u_{r+1} = (2+u_r)^2$ then, $u_2 = (2+u_1)^2$ $u_2 = (2+0)^2 = 4$; $u_3 = (2+4)^2 = 36$; and $u_4 = (2+36)^2 = 1444$.

3.

(a) $u_1 = 1$ $u_2 = \sqrt{(4-u_1)^2}$ $u_2 = \sqrt{(4-1)^2} = 3$ $u_3 = \sqrt{(4-3)^2} = 1$, $u_4 = \sqrt{(4-1)^2} = 3$, etc, alternately 1 and 3.
(b) $u_1 = 6$ $u_2 = \sqrt{(4-6)^2} = 2$ $u_3 = \sqrt{(4-2)^2} = 2$, therefore the first term is 6, then 2 repeats.
(c) The value of u_r for which all the terms of the sequence are equal to each other implies $u_{r+1} = u_r$; that is:

$$4 - u_r = u_r$$

$$4 = 2u_r \quad \texttt{or} \quad u_r = 2$$

4.

Given $u_{n+1} = u_n^2 - 1$:

(a) Behavior of the sequence for the case $u_1 = 0$:
$u_2 = 0^2 - 1 = -1$ $u_3 = (-1)^2 - 1 = 0$, $u_3 = 0^2 - 1 = -1$, that is, the terms of the sequence are 0 and -1, alternately.
Behavior of the sequence for the case $u_1 = 1$:
$u_2 = 1^2 - 1 = 0$ $u_3 = (0)^2 - 1 = -1$, $u_3 = (-1)^2 - 1 = 0$, that is, the terms of the sequence are 1, and then alternately 0 and -1.
Behavior of the sequence for the case $u_1 = 2$:
$u_2 = 2^2 - 1 = 3$ $u_3 = 3^2 - 1 = 8$, $u_3 = 8^2 - 1 = 63$, etc the terms of the sequence increase without bound.
(b) Given that $u_1 = u_2$ let's call them just u, then $u = u^2 - 1$, which leads to the quadratic equation $u^2 - u - 1 = 0$; by applying the quadratic formula we have that $u = \dfrac{1 \pm \sqrt{5}}{2}$. (c) Given that $u_3 = u_1$ considering that $u_3 = u_2^2 - 1$ then $u_1 = u_2^2 - 1$ also $u_2 = u_1^2 - 1$; then, substituting the latter result into the previous equation for u_1, we get:

$$u_1 = (u_1^2 - 1)^2 - 1$$

$$u_1 = u_1^2 - 2u_1^2 + 1 - 1$$

$$0 = u_1^2 - 2u_1^2 - u_1.$$

5.

The rth term of a arithmetic progression is $1 + 4r$; therefore, the first term is $a_1 = 1 + 4(1) = 5$. The sum of the first n terms of the progressions is given by $S_n = \dfrac{n}{2}(a_1 + l)$ `where` $l = 1 + 4r$ `then` $S_n = \dfrac{n}{2}(5 + 1 + 4r)$ or $S_n = \dfrac{n(6+4r)}{2}$ that is $S_n = n(2n+3)$.

6.

Sum of the first two terms: $18 = \dfrac{2}{2}(2a_0 + d)$; then, $18 = 2a_o + d$;
Sum of the first four terms, $52 = \dfrac{4}{2}(2a_0 + 3d)$; which simplifies to $26 = 2a_0 + 3d$; then, solving a system of two equation by substitution —solving from the first equation we get, $2a_0 = 18 - d$; that is:

$$26 = (18 - d) + 3d$$

$$d = \frac{26-18}{2} = 4$$

Substituting d value in $18 = 2a_o + d$, and solving for a_0:

$$a_0 = \frac{1}{2}(18-4) = 7$$

Knowing a_0 and d, we calculate the sum of the first eight terms:

$$S_8 = \frac{8}{2}(2a_o + 7d) = 4[2(7) + 7(4)] = 168.$$

7.

Sum of the first 20 terms is 50: $S_{20} = \dfrac{20}{2}(2a_0 + 19d) = 50$ or $2a_0 + 19d = 5$.
Sum of the next 20 terms is -50; therefore, $S_{40} = 0$; that is: $S_{40} = \dfrac{40}{2}(2a_0 + 39d) = 0$ or $2a_0 + 39d = 0$.

We need to solve a system of two simultaneous equations. Multiplying the first one by -1 we get:

$$-2a_0 - 19d = -5$$
$$2a_0 + 39d = 0$$

The sum of the equations, term by term, yields: $-20d = 5$; therefore $d = -\frac{5}{20} = -\frac{1}{4}$; once d value is known, we solve in one of the equations for a_0; that is:

$$2a_0 + 19(-\frac{1}{4}) = 5$$
$$2a_0 = 5 + \frac{19}{4}$$
$$a_0 = \frac{39}{8}$$

We are ready to calculate the sum of the first hundred terms:

$$S_{100} = \frac{100}{2}\left[2\left(\frac{39}{8} + 99(-\frac{1}{4}\right)\right] = -750.$$

8.

The sum of the first n terms: $S_n = \frac{n}{2}[2a + (n-1) - 1] = \frac{n}{2}(2a - n + 1)$.
The sum of the first $3n$ terms: $S_{3n} = \frac{3n}{2}[2a + (3n-1) - 1] = \frac{3n}{2}(2a - 3n + 1)$.
These two sums are equal to one another, therefore:

$$\frac{n}{2}(2a - n + 1) = \frac{3n}{2}(2a - 3n + 1).$$

Multiplying by 2, diving by n, we obtain:

$$2a - n + 1 = 6a - 9n + 3$$
$$9n - n + 1 - 3 = 6a - 2a$$
$$8n - 2 = 4a$$

$4n - 1 = 2a$ then, a in terms of n is: $a = \dfrac{4n-1}{2}$

9.

The given arithmetic progression has $d = 3$; the last term, l is 1000; then, the number of terms is given by: $l = a_1 + (n-1)d$ substituting in the values we have: $1000 = 1 + 3n - 3$, therefore $n = 334$; the sum of the progression is given by:

$$S_{1000} = \frac{n}{2}(a_1 + l) = \frac{334}{2}(1 + 1000) = 167167.$$

The removed sequence, " every third terms" is: 7, 16, ...; let's find the sum of such sequence, and subtract the resulting sum from the sum of the initial sequence. The removed sequence has difference $d = 9$ and a first term $a = 7$; the last term 997 (terms being removed minus 7 –the first term– are multiples of 9; which is given by the fact that each nth term is the sequence is given by $a_n = a_1 + (n-1)9$. Then, $997 = 7 + 9n - 9$ or, $n = 111$:

$$S_{111} = \frac{111}{2}(7 + 997) = 55722$$

Therefore, after removing each $3rd$ term of the sequence, the resulting sequence sum is: $S = 167167 - 55722 = 111445$.

10.
We know that, for an arithmetic progression $S = \frac{n}{2}(2a_1 + (n-1)d)$.
Sum is T, difference is d; then, $T = \dfrac{100}{2}(2a + 99d)$=$50(2a + 99d)$;
Sum of odd numbers terms has a common difference is $2d$, and the total sum is, according the the question $\frac{1}{2} - 1000$; the 50 odd numbers, $n = 50$; now we have:

$$\frac{1}{2}T - 1000 = \frac{50}{2}(2a + 49(2d))$$

Multiplying by 2, we obtain:

$$T - 2000 = 50(2a + 98d)$$

Recall that $T = 500(2a + 99d)$, substituting T in the previous result we get:

$$50(2a + 99d) - 2000 = 50(2a + 98d)$$

That is, $d = 40$.

11.

Given, $a = 1$, $d = 0.1$ $l = 100$; knowing that, the last term is given by: $l = a_1 + (n-1)d$; then,

(a)

$$100 = 1 + (n-1)0.1$$
$$100 = 1 + 0.1n - 0.1$$
$$n = 991$$

(b)

$$S = \frac{n}{2}(a_1 + l) = \frac{991}{2}(1 + 100) = 50045.5$$

12.

(a) $u_3 = 2(3)^2 = 18$
(b) $u_n = 2n^2$ and $u_{n+1} = 2(n+1)^2$

$$u_{n+1} - u_n = 2(n+1)^2 - 2n^2$$
$$u_{n+1} - u_n = 4n + 2$$

(c) $u_1 = 2$; $u_2 = 8$; $u_3 = 18$; $u_4 = 32$; the arithmetic sequence of the differences between successive terms is: 2,8,18,32... That is, $a_1 = 6$ and $d = 4$; the sum of the first 1000 terms is: $S_{1000} = \frac{1000}{2}[2(6) + 999(4)] = 200400$.

13.

280,288,... Then, $a = 280 \quad d = 8$;

$$l = a_1 + (n-1)d$$

the 91*st* term is:——

$$l_{91} = 280 + 90(8) = 1000.$$

The sum of the first 91 terms is:

$$S_{91} = \frac{91}{2}(280 + 1000) = 58240.$$

From week 91 to 104, 13 weeks, the production remains at 1000 per week, for a total, in this period, of 13000; the grand total is: $S_{91} + 13000 = 71240$.

14.

28£ per year for the first 21 years, increased by 14£ and the end of each 21 years period; in 999 years there are $\frac{999}{21} = 47$ remainder, 12. Therefore,
(a) There are 47 complete 21 years period, 12 years left over.
(b) The sum of the sequence is given by:

$$S_{47} = (28 \cdot 21) + (42 \cdot 21) + (56 \cdot 21) + ...$$

21 is a factor for each term of the sum:

$$21S_{47} = 28 + 42 + 56 + ...$$
$$21S_{47} = 21 \cdot \frac{47}{2}[2(28) + (47-1)14] = 21 \cdot 16450 = 345450.$$

15.

First term is a; difference is $d = 10$.

$$Sn = \frac{n}{2}(2a + (n-1)10) = 10000$$
$$n(2a + 10n - 10) = 20000$$
$$2a + 10n - 10 = \frac{20000}{n}$$
$$2a = \frac{20000}{n} + 10 - 10n$$
$$a = \frac{10000}{n} - 5n + 5$$
$$a = \frac{10000}{n} - 5(n-1)$$

The nth term of the sequence is given by l_n:

$$l_n = a + (n-1)10$$

Substituting the previous result for a into the l_n term equation:
The nth is equal to:

$$= \left(\frac{10000}{n} - 5n + 5\right) + 10n - 10$$

$$= \frac{1000}{n} + 5(n-1)$$

The nth term is less than 500; that is:

$$\frac{1000}{n} + 5(n-1) < 500$$

$$10000 + 5n^2 - 5n < 500n$$

$$2000 + n^2 - n - 100n < 0$$

$$n^2 - 101n + 2000 < 0$$

By applying the quadratic formula, ($a = 1,\ b = -101, and\ c = 2000$) we obtain that the largest solution for n is 73.96; and, since n is an integer, the largest possible value for n is 73.