Practice 10a

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Suggested solutions for exercises 7C —from number 6 to 16—, page 110 of Pure Mathematics 1 by Hugh Neil and Douglas Qualing.

6.

$y = 5 + \frac{1}{5}S - \frac{1}{800}S^2$. The derivative of this function represents the change in fuel efficiency with respect to speed (S).

So, $\frac{dy}{dS} = \frac{1}{5} - \frac{S}{400}$

Equate the the previous result to zero, in order to find at which value of the independent variable, in this case S, the function reaches its maximum:

$$\frac{dy}{dS} = \frac{1}{5} - \frac{S}{400} = 0$$

Multiplying the equation by 400 throughout:

$$80 - S = 0$$

$$S = 80$$

The speed at which the car should be driven for maximum economy is 80 km/h.

7.

Equation of height: $h = 20t - 5t^2$. Velocity is the change of distance with respect to time, $\frac{dh}{dt} = 20 - 10t$. At maximum height, instantly, the velocity is equal to zero, then: $\frac{dh}{dt} = 20 - 10t = 0$; therefore, $t = 2$. Considering the equation of the height, at $t = 2$, the height is $h(2) = 20(2) - 5(2^2) = 20m$.

8.

Notice: in the problems that follow, there is a function to maximise or minimise, and another equation that establishes a constrain for the given function. In all cases the function to optimize contains two variables, the constrain equation allows us to write the function as a function of one variable; then we take the first derivative of the function, and, by equating the equation of the derivative to zero, we are able to find the maximum or minimum value of the function in the given domain:

The function to optimize,

$$f(x) = x \cdot y$$

The constrain equation:

$$x + y = 12 \texttt{ or } y = 12 - x$$

The original function becomes:

$$f(x) = x(12 - x)$$

$$12 - 2x = 0 \quad \texttt{or} \quad x = 6$$

Considering the constrain equation $x + y = 12$, and solving for y we get $y = 6$.
Therefore the maximum value of the function $f(x) = 6 \cdot 6 = 36$.

9.

The function to maximise is $f(x) = x \cdot y$; the constrain equation is $x \cdot y = 20$; hence, $y = \frac{20}{x}$. The original function becomes, $f(x) = x + \frac{20}{x}$, then:

$$\frac{df}{dx} = 1 - \frac{20}{x^2}$$
$$0 = 1 - \frac{20}{x^2}$$
$$0 = x^2 - 20$$
$$x = \pm\sqrt{20}$$

But we are looking for a "positive Real number", so our solution is $\sqrt{20}$.
$y = \frac{20}{x}$; or $y = \frac{20}{\sqrt{20}} = \sqrt{20}$.
Since $\sqrt{20}$ simplifies to $\sqrt{4 \cdot 5} = 2\sqrt{5}$; then, $2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$.

10.

The function is given by $V = \pi r^2 h$; the constrain equation is $r + h = 6$, or, $h = 6 - r$, therefore:

$$V = \pi r^2(6 - r) = 6\pi r^2 - \pi r^3$$
$$\frac{dV}{dr} = 12\pi r - 3\pi r^2$$
$$12\pi r - 3\pi r^2 = 0$$
$$3\pi r(4 - r) = 0$$

Therefore, the solutions are, $r = 0$ and $r = 4$. The " minimum" value, $r = 0$, is trivial; the maximum value, at $r = 4$ $h = 6 - 4 = 2$, hence the maximum volume will be:

$$V = \pi(4^2)(2) = 32\pi$$

11.

Rectangle, two equal parallel sides, x; and two other equal parallel sides y. The perimeter is one metre. The equation is: $2x + 2y = 1$. The area function for a rectangle is $A = x \cdot y$; the constrain is $2x + 2y = 1$, where $y = \frac{1}{2}(1 - 2x)$ therefore the function to maximise becomes:

$$A = x \cdot \frac{1}{2}(1 - 2x) = \frac{1}{2}x - x^2$$
$$\frac{dA}{dx} = \frac{1}{2} - 2x$$
$$\frac{1}{2} - 2x = 0 \ \texttt{or} \ x = \frac{1}{4} = 0.25m$$

12.

One side of the rectangular sheep pen is a hedge; let's label the parallel side as x. The two other parallel sides to be fenced are labeled y. Since 120 metres of fencing are available, $x + 2y = 120$. The area of the rectangle is given by $A = x \cdot y$. From the previous equation $y = \frac{1}{2}(120 - x)$. Then, the area of the rectangle is:

$$A = x \cdot \frac{1}{2}(120 - x) = \frac{1}{2}x(120 - x).$$

This result is equivalent to:

$$A = 60x - \frac{1}{2}x^2$$

Whose derivative is:

$$\frac{dA}{dx} = 60 - x$$

$$60 - x = 0 \ \texttt{ or } \ x = 60$$

Therefore, we have found that $x = 60$ metres is the length of the side that maximise the area.

$$A(60) = \frac{1}{2}60(120 - 60) = 1800m^2$$

13.

Equal squares of sides x are cut. The shortest side measures 40 cm; since a cut is made from each corner the maximum size for x is 20; the minimum, any number greater than zero is a cut. The domain of of x (the permissible values for x) are: $0 < x < 20$.

Formula for the volume: a cut of size x is made from each corner, so each side is reduced by $2x$. And because the the sheet is folded up, the high of the box is just x; then, the equation of the volume is:

$$V = (50 - 2x)(40 - 2x)x$$

$$V = (2000 - 100x - 80x + 4x^2)x$$

$$V = 2000x - 180x^2 + 4x^3$$

The derivative of the volume with respect to the side x is:

$$V = 2000 - 360x + 12x^2$$

Which is equivalent to:

$$V = 500 - 90x + 3x^2$$

A quadratic equation whose solutions are: $x_1 = 22.64$ and $x_2 = 7.36$. The first result is discarded since it is out the domain of the function. The answers is $x = 7.36$ cm.

14.

This is an open rectangular box. Open, it means has a square base whose surface area (SA) is side squared or s^2 and four sides of are sh, h denoting the height. The surface area function to minimise is:

$$SA = f(x) = s^2 + 4sh$$

The constrain equation is given by the volume sought $V = 4000\ cm^3 = s^2 \cdot h$ which implies that $h = \dfrac{4000}{s^2}$; then,

$$SA = f(x) = 4s\left(\frac{4000}{s^2}\right) + s^2$$

Which is equivalent to:

$$f(x) = \frac{4000}{s} + s^2$$

Deriving the equation, and equating the result to zero:

$$f(x) = displaystyle - \frac{16000}{s^2} + 2s = 0$$

Multiplying the previous equation by s^2:

$$-16000 + 2s^3 = 0$$

$$s = \sqrt[3]{\frac{16000}{2}} = \sqrt[3]{8000} = 20\ cm.$$

The length of the side that minimise the material to construct a box of $V = 4000cm^3$ is $20\ cm$.

15.

An open cylinder has surface area $SA = f(x) = 2\pi rh + \pi r^2$. The surface area is $5000cm^3$, then, $5000 = 2\pi rh + \pi r^2$ or $h = \dfrac{5000 - \pi r^2}{2\pi r}$. The volume of the cylinder, given by, $V = \pi r^2 h$ is the function we seek to maximise. Substituting the previous result for the height into the volume function it yields:

$$V = \pi r^2\left(\frac{5000 - \pi r^2}{2\pi r}\right)$$

Which simplifies to:

$$V = \frac{1}{2}r(5000 - \pi r^2)$$

This result is equivalent to:

$$V = 2500r - \frac{1}{2}\pi r^3$$

The derivative of this function, the change of volume with respect to he radius is:

$$\frac{dV}{dr} = 2500 - \frac{3}{2}\pi r^2$$

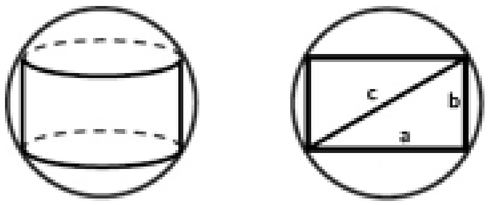
$$2500 - \frac{3}{2}\pi r^2 = 0$$

$$r = \sqrt{\frac{5000}{3\pi}} = 23.03\ cm$$

The maximum possible capacity of the bin for a radius of $23.03\ cm$ is:

$$V = \frac{1}{2}(23.03)[5000 - \pi(23.03)^2] = 38,388\ cm^3$$

16.



The radius of the sphere is equal to 10 cm; therefore, the diagonal $c = 20\ cm$. a is twice the diameter of the cylinder or $= 2r$ (r denotes the radius of the cylinder). b is the hight of the cylinder, denoted by h. Hence,by the Pythagorean theorem:

$$20^2 = (2r)^2 + h^2$$
$$400 = 4r^2 + h^2$$

Solving for the radius of the cylinder:

$$r^2 = \frac{400 - h^2}{4}$$

The previous result is the equation of the constrain; the volume of the cylinder is given by:

$$V = \pi r^2 h$$

The composition of the functions yields:

$$V = \frac{\pi}{4}(400h - h^3)$$

...whose derivative is:

$$\frac{dV}{dh} = \frac{\pi}{4}(400 - 3h^2)$$

It implies that,

$$3h^2 = 400$$
$$h = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}}$$

Knowing the height at which the volume of the cylinder is maximum, we calculate the radius r and then the volume:

$$r^2 = \frac{400 - h^2}{4} = \frac{400 - (\frac{20}{\sqrt{3}})^2}{4} = \frac{200}{3}$$

The maximum possible volume of the cylinder is reached when $r^2 = \dfrac{200}{3}$ and $h = \dfrac{20}{\sqrt{3}}$:

$$V = \pi\left(\frac{200}{3}\right)\left(\frac{20}{\sqrt{3}}\right) = 2418\ cm^3$$