Miscellaneous exercise 12

1 Differentiate $(4x - 1)^20$ with respect to $x$.

2 Differentiate $\frac{1}{(3-4x)^2}$ with respect to $x$.

3 Differentiate $2(x^4 + 3)^5$ with respect to $x$.

4 Find the equation of the tangent to the curve $y = (x^2 - 5)^6$ at the point $(2,1)$.

5 Given that $y = \sqrt{x^3 + 1}$, show that $\frac{dy}{dx} > 0$ for all $x > -1$.

6 Given that $y = \frac{1}{2x-1} + \frac{1}{(2x-1)^2}$, find the exact value of $\frac{dy}{dx}$ when $x = 2$.

7 Find the equation of the tangent to the curve $y = (4x + 3)^5$ at the point $(-\frac{1}{2},1)$, giving your answer in the form $y = mx + c$.

8 Find the coordinates of the stationary point of the curve with equation $y = \frac{1}{x^2 + 4}$.

9 Find the equation of the normal to the curve $y = \sqrt{2x^2 + 1}$ at the point $(2,3)$.

10 The radius of a circular disc is increasing at a constant rate of 0.003 cm s$^{-1}$. Find the rate at which the area is increasing when the radius is 20 cm.

11 A viscous liquid is poured on to a flat surface. It forms a circular patch whose area grows at a steady rate of 5 cm$^2$ s$^{-1}$. Find, in terms of $\pi$,
(a) the radius of the patch 20 seconds after pouring has commenced,
(b) the rate of increase of the radius at this instant.
12 Find the equation of the tangent to the curve \( y = \frac{50}{(2x-1)^2} \) at the point \((3,2)\), giving your answer in the form \( ax + by + c = 0 \), where \( a \), \( b \) and \( c \) are integers.

13 Sketch the graph of \( y = (x - 2)^2 - 4 \) showing clearly on your graph the coordinates of any stationary points and of the intersections with the axes.
Find the coordinates of the stationary points on the graph of \( y = (x - 2)^3 - 12(x - 2) \) and sketch the graph, giving the exact coordinates (in surd form, where appropriate) of the intersections with the axes.

14 Differentiate \( \sqrt{x + \frac{1}{x}} \) with respect to \( x \).

15 The formulae for the volume of a sphere of radius \( r \) and for its surface area are \( V = \frac{4}{3} \pi r^3 \) and \( A = 4 \pi r^2 \) respectively. Given that, when \( r = 5 \) m, \( V \) is increasing at a rate of \( 10 \) m\(^3\) s\(^{-1}\), find the rate of increase of \( A \) at this instant.

16 Using differentiation, find the equation of the tangent at the point \((2,1)\) on the curve with equation \( y = \sqrt{x^2 - 3} \).

17 Differentiate \( \frac{1}{(3t^2 + 5)^2} \) with respect to \( t \).

18 (a) Curve \( C_1 \) has equation \( y = \sqrt{4x - x^2} \). Find \( \frac{dy}{dx} \) and hence find the coordinates of the stationary point.
(b) Show that the curve \( C_2 \) with equation \( y = \sqrt{x^2 - 4x} \) has no stationary point.

19 A curve has equation \( y = \frac{1}{12} (3x+1)^4 - 8x \).
(a) Show that there is a stationary point where \( x = \frac{1}{3} \) and determine whether this stationary point is a maximum or a minimum.
(b) At a particular point of the curve, the equation of the tangent is \( 48x + 3y + c = 0 \). Find the value of the constant \( c \).

20 If a hemispherical bowl of radius 6 cm contains water to a depth of \( x \) cm, the volume of the water is \( \frac{1}{3} \pi x^2(18 - x) \). Water is poured into the bowl at a rate of \( 3 \) cm\(^3\) s\(^{-1}\). Find the rate at which the water level is rising when the depth is 2 cm.

21 An underground oil storage tank \( ABCDEFGH \) is part of an inverted square pyramid, as shown in the diagram. The complete pyramid has a square base of side 12 m and height 18 m. The depth of the tank is 12 m.

When the depth of oil in the tank is \( h \) metres, show that the volume \( V \) m\(^3\) is given by \( V = \frac{4}{27} (h + 6)^3 - 32 \).

Oil is being added to the tank at the constant rate of \( 4.5 \) m\(^3\) s\(^{-1}\). At the moment when the depth of oil is 8 m, find the rate at which the depth is increasing.
22 A curve has equation \( y = (x^2 - 1)^3 - 3(x^2 - 1)^2 \). Find the coordinates of the stationary points and determine whether each is a minimum or a maximum. Sketch the curve.

23 Find the coordinates of the stationary point of the curve \( y = \frac{1}{2x + 1} - \frac{1}{(2x + 1)^2} \) and determine whether the stationary point is a maximum or a minimum.

24 Find the coordinates of the stationary point of the curve \( y = \sqrt{4x - 1} + \frac{9}{\sqrt{4x - 1}} \) and determine whether the stationary point is a maximum or a minimum.

25 (a) Expand \((ax + b)^3\) using the binomial theorem. Differentiate the result with respect to \(x\) and show that the derivative is \(3a(ax + b)^2\).

(b) Expand \((ax + b)^4\) using the binomial theorem. Differentiate the result with respect to \(x\) and show that the derivative is \(4a(ax + b)^3\).

(c) Write down the expansion of \((ax + b)^n\) where \(n\) is a positive integer. Differentiate the result with respect to \(x\). Show that the derivative is \(na(ax + b)^{n-1}\).