Assignment 13, Vectors: exercise 13D: 10-16 page 207

imathesis.com
By Carlos Sotuyo

Suggested solutions for selected exercises 13D, problems 12-16, page 207 from Pure Mathematics 1, by Hugh Neil and Douglas Quailing, Cambridge University Press, 2002.

12. Find the angles between the line joining $(1, 3, -2)$ and $(2, 5, -1)$ and the line joining $(-1, 4, 3)$ to $(3, 2, 1)$.

Let's label our points:A, $(1, 3, -2)$; B, $(2, 5, -1)$; C, $(-1, 4, 3)$ and D, $(3, 2, 1)$. Then, position vector $\vec{AB}$ is $(2-1, 5-3, -1-(-2)) = (1, 2, 1)$; similarly, $\vec{CD} = (4, -2, -2)$. The angle between the vectors is given by the dot product:

$\vec{AB} \cdot \vec{CD} = |\vec{AB}| \cdot |\vec{CD}| \cos\theta$ and, considering that $|\vec{AB}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$ and $|\vec{CD}| = \sqrt{(4)^2 + (-2)^2 + (-2)^2} = \sqrt{24}$
Substituting into the dot product formula:
$(1 \cdot 4 + 2 \cdot -2 + 1 \cdot -2) = -2$; therefore,

$$\theta = \cos^{-1}\left(\frac{\vec{AB} \cdot \vec{CD}}{|\vec{AB}| \cdot |\vec{CD}|}\right) = \cos^{-1}\left(\frac{-2}{\sqrt{6} \cdot \sqrt{24}}\right) = 99.6°$$

If, instead, the position vector is $\vec{DC} = (-4, 2, 2)$, the dot product $\vec{AB} \cdot \vec{DC} = 2$, so the acute angle is given by:

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6} \cdot \sqrt{24}}\right) = 80.40°$$

13.

Find the angles between the diagonals of a cube.

Let's position the cube on the xy plane, so the vertices on the xy plane are: $A, (0, 0, 0)$; $B, (0, 1, 0)$; $C, (-1, 1, 0)$ and $D, (-1, 0, 0)$.
The top vertices are: $E, (0, 0, 1)$; $F, (0, 1, 1)$; $G, (-1, 1, 1)$ and $H, (-1, 0, 1)$.
The diagonals position vectors are: $\vec{AG} = (-1, 1, 1)$ and $\vec{EC} = (-1, 1, -1)$.
The angle between the two vector is given by the dot product formula:
$\vec{AG} \cdot \vec{EC} = |\vec{AG}| \cdot |\vec{EC}| \cos\theta$

$1 = \sqrt{3} \cdot \sqrt{3}\ cos\theta$
$\theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.53°$

14.

$ABCD$ is the base of a square pyramid of side 2 units, and V is the vertex. The pyramid is symmetrical, and of height 4 units. Calculate the acute angle between AV and BC, giving your answers in degrees correct to one decimal place.

$A = (0, 0, 0)$ $D = (-2, 0, 0)$ $V = (-1, 1, 4)$

Since $BC \parallel AD$,the angle between AV and BC is equal to the angle between AV and AD. Position vectors are: $\vec{AD} = (-2, 0, 0)$ and $\vec{AV} = (-1, 1, 4)$; the magnitude of those vector are, respectively, 2 and $\sqrt{18}$; therefore, the angle is given by:
$\vec{AV} \cdot \vec{AD} = |\vec{AV}| \cdot |\vec{AD}| \cos\theta$

$2 = 2\sqrt{18}\ cos\theta$
$\theta = \cos^{-1}\left(\frac{2}{2\sqrt{18}}\right) = 76.37°$

15.

Two airplanes are flying in directions given by the vectors $300i + 400j + 2k$ and $-100i + 500j - k$. A person from the flight control centre is plotting their paths on a map. Find the acute angle between their paths on the map.

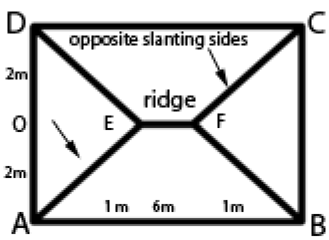
Given, $\vec{u} = 300i + 400j + 2k$ and $\vec{v} = -100i + 500j - k$; then, $\vec{u} \cdot \vec{v} = 170002$ and $|\vec{u}| = 500.004$; $|\vec{v}| = 5009.903$; therefore,

$$\theta = \cos^{-1}\left(\frac{170002}{(500.004)(509.903)}\right) = 48.2°$$

16.

The roof of a house has a rectangular base of side 4 metres by 8 metres. The ridge line of the roof is 6 metres long, and centred 1 metre above the base of the roof. Calculate the acute angle between two opposite slanting edges of the roof.

Graph not drawn to scale: observe that AD is located along the x-axis; the origin, point O, $(0, 0, 0)$ divides AD into two equal segments. The line of the ridge would be on the y-axis:



The coordinates of interest are: $A = (2, 0, 0)$; $C = (-2, 8, 0)$; $E = (0, 1, 1)$; and $F = (0, 7, 1)$; Therefore,

$\vec{AE} = (-2, 1, 1)$; and $\vec{FC} = (-2, 1, 11)$ then, $\vec{AE} \cdot \vec{FC} = (4 + 1 - 1) = 4$ while $|\vec{AE}| = \sqrt{6}$; $|\vec{FC}| = \sqrt{6}$,

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{6}\,\sqrt{6}}\right) = 48.18° = 48.2°$$