Math 211-04
4–30–2007

Review Problems for the Final

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Find the area of the intersection of the interiors of the circles

$$x^2 + (y-1)^2 = 1 \quad \text{and} \quad (x-\sqrt{3})^2 + y^2 = 3.$$

2. Does the series

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$$

converge absolutely, converge conditionally, or diverge?

3. Find the sum of the series

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots.$$

4. In each case, determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}$.

(b) $\displaystyle\frac{2}{1} + \frac{2\cdot 5}{1\cdot 5} + \cdots + \frac{2\cdot 5\cdot \cdots \cdot (3n-1)}{1\cdot 5\cdot \cdots \cdot (4n-3)}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$.

(d) $\displaystyle\frac{2}{3} - \frac{5}{8} + \frac{8}{13} - \frac{11}{18} + \cdots$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4n+2}{\sqrt{n^5+16}}$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\cos(e^n)}{n}$.

5. Find the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(x-5)^n$$

converges absolutely.

6. Compute the following integrals.

(a) $\displaystyle\int e^x \cos 2x \, dx$.

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}} \, dx$.

(c) $\displaystyle\int \frac{5x^2-6x-5}{(x-1)^2(x+2)}\,dx.$

(d) $\displaystyle\int (\sin 4x)^3(\cos 4x)^2\,dx.$

(e) $\displaystyle\int (\sin 4x)^2(\cos 4x)^2\,dx.$

(f) $\displaystyle\int \frac{1}{(-3-4x-x^2)^{3/2}}\,dx.$

7. Let R be the region bounded above by $y=x+2$, bounded below by $y=-x^2$, and bounded on the sides by $x=-2$ and by the y-axis. Find the volume of the solid generated by revolving R about the line $x=1$.

8. Compute $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+8x}-x\right).$

9. Compute $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}.$

10. If $x=t+e^t$ and $y=t+t^3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t=1$.

11. (a) Find the Taylor expansion at $c=1$ for e^{2x}.

(b) Find the Taylor expansion at $c=1$ for $\dfrac{1}{3+x}$. What is the interval of convergence?

12. Find the area of the region which lies between the graphs of $y=x^2$ and $y=x+2$, from $x=1$ to $x=3$.

13. Find the area of the region between $y=x+3$ and $y=7-x$ from $x=0$ to $x=3$.

14. The base of a solid is the region in the x-y-plane bounded above by the curve $y=e^x$, below by the x-axis, and on the sides by the lines $x=0$ and $x=1$. The cross-sections in planes perpendicular to the x-axis are squares with one side in the x-y-plane. Find the volume of the solid.

15. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n(2^n)^3}.$

16. Find the slope of the tangent line to the polar curve $r=\sin 2\theta$ at $\theta=\dfrac{\pi}{6}$.

17. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.

18. Let

$$x=\frac{\sqrt{3}}{2}t^2,\quad y=t-\frac{1}{4}t^3.$$

Find the length of the arc of the curve from $t=-2$ to $t=2$.

19. Find the area of the surface generated by revolving $y=\dfrac{1}{3}x^3$, $0\le x\le 2$, about the x-axis.

20. (a) Convert $(x-3)^2+(y+4)^2=25$ to polar and simplify.

(b) Convert $r=4\cos\theta-6\sin\theta$ to rectangular and describe the graph.

21. Find the area of the region inside the cardioid $r=1+\cos\theta$ and outside the circle $r=3\cos\theta$.

Solutions to the Review Problems for the Final

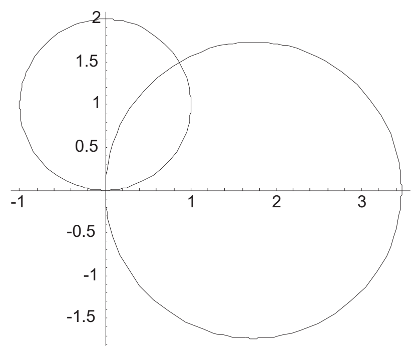
1. Find the area of the intersection of the interiors of the circles

$$x^2 + (y-1)^2 = 1 \quad \text{and} \quad (x-\sqrt{3})^2 + y^2 = 3.$$

Convert the two equations to polar:

$$x^2 + y^2 - 2y + 1 = 1, \quad x^2 + y^2 = 2y, \quad r^2 = 2r\sin\theta, \quad r = 2\sin\theta.$$

$$(x-\sqrt{3})^2 + y^2 = 3, \quad x^2 - 2\sqrt{3}x + 3 + y^2 = 3, \quad x^2 + y^2 = 2\sqrt{3}x, \quad r^2 = 2\sqrt{3}r\cos\theta, \quad r = 2\sqrt{3}\cos\theta.$$



Set the equations equal to solve for the line of intersection:

$$2\sin\theta = 2\sqrt{3}\cos\theta, \quad \tan\theta = \sqrt{3}, \quad \theta = \frac{\pi}{3}.$$

The region is "orange-slice"-shaped, with the bottom/right half bounded by $r = 2\sin\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{3}$ and the top/left half bounded by $r = 2\sqrt{3}\cos\theta$ from $\theta = \dfrac{\pi}{3}$ to $\theta = \dfrac{\pi}{2}$. Hence, the area is

$$A = \int_0^{\pi/3} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\sqrt{3}\cos\theta)^2\,d\theta = 2\int_0^{\pi/3} (\sin\theta)^2\,d\theta + 6\int_{\pi/3}^{\pi/2} (\cos\theta)^2\,d\theta =$$

$$\int_0^{\pi/3} (1-\cos 2\theta)\,d\theta + 3\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\,d\theta = \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + 3\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} =$$

$$\frac{5}{6}\pi - \sqrt{3} \approx 0.88594. \quad \square$$

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2. Does the series

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$$

converge absolutely, converge conditionally, or diverge?

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{n^3+1}.$$

The absolute value series is

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}.$$

Since

$$\frac{n^2}{n^3+1} \approx \frac{1}{n}$$

for large values of n, I'll compare the series to $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

$$\lim_{n\to\infty} \frac{\dfrac{n^2}{n^3+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1.$$

The limit is finite ($\neq \infty$) and positive (> 0). The harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges. By Limit Comparison, the series $\sum_{n=1}^{\infty} \dfrac{n^2}{n^3+1}$ diverges. Hence, the original series does not converge absolutely.

Returning to the original series, note that it alternates, and

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0.$$

Let $f(n) = \dfrac{n^2}{n^3+1}$. Then

$$f'(n) = \frac{n(2-n^3)}{(1+n^3)^2} < 0$$

for $n > 1$. Therefore, the terms of the series decrease for $n \geq 2$, and I can apply the Alternating Series Rule to conclude that the series converges. Since it doesn't converge absolutely, but it *does* converge, it converges conditionally. □

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3. Find the sum of the series

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots.$$

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots = \frac{5}{9}\left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots\right) = \frac{5}{9} \cdot \frac{1}{1 - \left(-\dfrac{1}{3}\right)} = \frac{5}{9} \cdot \frac{3}{4} = \frac{5}{12}. \quad \square$$

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4. In each case, determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}$.

Apply the Integral Test. The function $f(n) = \dfrac{1}{n(\ln n)^{4/3}}$ is positive and continuous on the interval $[2, +\infty)$.

Since

$$f'(n) = -\frac{4}{3n^2(\ln n)^{7/3}} - \frac{1}{n^2(\ln n)^{4/3}},$$

it follows that $f'(n) < 0$ for $n \geq 2$. Hence, f decreases on the interval $[2, +\infty)$. The hypotheses of the Integral Test are satisfied.

Compute the integral:

$$\int_2^\infty \frac{1}{n(\ln n)^{4/3}}\,dn = \lim_{p\to\infty}\int_2^p \frac{1}{n(\ln n)^{4/3}}\,dn =$$

$$\lim_{p\to\infty}\left[-3\frac{1}{(\ln n)^{1/3}}\right]_2^p = -3\lim_{p\to\infty}\left(\frac{1}{(\ln p)^{1/3}} - \frac{1}{(\ln 2)^{1/3}}\right) = \frac{3}{(\ln 2)^{1/3}}.$$

(To do the integral, I substituted $u = \ln n$, so $du = \dfrac{1}{n}\,dn$.)

Since the integral converges, the series converges, by the Integral Test. □

(b) $\dfrac{2}{1} + \dfrac{2\cdot 5}{1\cdot 5} + \cdots + \dfrac{2\cdot 5\cdot\cdots\cdot(3n-1)}{1\cdot 5\cdot\cdots\cdot(4n-3)}$.

Apply the Ratio Test. The n^{th} term of the series is

$$a_n = \frac{2\cdot 5\cdot\cdots\cdot(3n-1)}{1\cdot 5\cdot\cdots\cdot(4n-3)},$$

so the $(n+1)$-st term is

$$a_{n+1} = \frac{2\cdot 5\cdot\cdots\cdot(3n-1)\cdot(3(n+1)-1)}{1\cdot 5\cdot\cdots\cdot(4n-3)\cdot(4(n+1)-3)}.$$

Hence,

$$\frac{a_{n+1}}{a_n} = \frac{2\cdot 5\cdot\cdots\cdot(3n-1)\cdot(3(n+1)-1)}{1\cdot 5\cdot\cdots\cdot(4n-3)\cdot(4(n+1)-3)}\cdot\frac{1\cdot 5\cdot\cdots\cdot(4n-3)}{2\cdot 5\cdot\cdots\cdot(3n-1)} = \frac{3(n+1)-1}{4(n+1)-3} = \frac{3n+2}{4n+1}.$$

The limiting ratio is

$$\lim_{n\to\infty}\frac{3n+2}{4n+1} = \frac{3}{4}.$$

The limit is less than 1, so the series converges, by the Ratio Test. □

(c) $\displaystyle\sum_{n=1}^\infty \left(1+\frac{1}{n}\right)^{-n^2}$.

Apply the Root Test.

$$a_n^{1/n} = \left(1+\frac{1}{n}\right)^{-n}.$$

The limit is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^n\right\}^{-1} = \left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right\}^{-1} = e^{-1}.$$

Since $e^{-1} = \dfrac{1}{e} < 1$, the series converges, by the Root Test. □

(d) $\dfrac{2}{3} - \dfrac{5}{8} + \dfrac{8}{13} - \dfrac{11}{18} + \cdots$.

Since

$$\lim_{n\to\infty}\frac{2+3n}{3+5n} = \frac{3}{5},$$

it follows that $\lim_{n\to\infty} a_n$ is undefined — the values oscillate, approaching $\pm\dfrac{3}{5}$. Since, in particular, the limit is nonzero, the series diverges, by the Zero Limit Test. □

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4n+2}{\sqrt{n^5+16}}$.

Apply Limit Comparison:

$$\lim_{n\to\infty} \frac{\dfrac{3n^2+4n+2}{\sqrt{n^5+16}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3n^{5/2}+4n^{3/2}+2n^{1/2}}{\sqrt{n^5+16}} = 3.$$

The limit is finite and positive. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, because it's a p-series with $p = \dfrac{1}{2} < 1$. Therefore, the original series diverges by Limit Comparison. □

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\cos(e^n)}{n}$.

$$\begin{array}{ccccc} -1 & \le & \cos(e^n) & \le & 1 \\ 4 & \le & 5+\cos(e^n) & \le & 6 \\ \dfrac{4}{n} & \le & \dfrac{5+\cos(e^n)}{n} & \le & \dfrac{6}{n} \end{array}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{n}$ diverges, because it's 4 times the harmonic series. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5+\cos(e^n)}{n}$ diverges by Direct Comparison. □

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5. Find the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(x-5)^n$$

converges absolutely.

Apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{((n+1)!)^2}{(2(n+1))!}|x-5|^{n+1}}{\dfrac{(n!)^2}{(2n)!}|x-5|^n} = \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!}\frac{|x-5|^{n+1}}{|x-5|^n} = \frac{(n+1)^2}{(2n+1)(2n+2)}|x-5|.$$

The limiting ratio is

$$\lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)}|x-5| = \frac{1}{4}|x-5|.$$

The series converges absolutely for $\dfrac{1}{4}|x-5| < 1$, i.e. for $1 < x < 9$. The series diverges for $x < 1$ and for $x > 9$.

You'll probably find it difficult to determine what is happening at the endpoints! However, if you experiment — compute some terms of the series for $x = 9$, for instance — you'll see that the individual terms are growing larger, so the series at $x = 1$ and at $x = 9$ diverge, by the Zero Limit Test. □

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6. Compute the following integrals.

(a) $\int e^x \cos 2x \, dx$.

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & e^x & & \cos 2x \\ & & \searrow & \\ - & e^x & & \dfrac{1}{2}\sin 2x \\ & & \searrow & \\ + & e^x & \rightarrow & -\dfrac{1}{4}\cos 2x \end{array}$$

$$\int e^x \cos 2x \, dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x \, dx,$$

$$\frac{5}{4}\int e^x \cos 2x \, dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x,$$

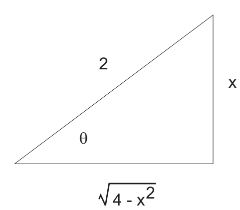
$$\int e^x \cos 2x \, dx = \frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x + C. \quad \square$$

(b) $\int \frac{x^2}{\sqrt{4-x^2}} \, dx$.

$$\int \frac{x^2}{\sqrt{4-x^2}} \, dx = \int \frac{4(\sin\theta)^2}{\sqrt{4-4(\sin\theta)^2}} 2\cos\theta \, d\theta = \int \frac{4(\sin\theta)^2}{\sqrt{4(\cos\theta)^2}} 2\cos\theta \, d\theta = 4\int (\sin\theta)^2 \, d\theta =$$

$$[x = 2\sin\theta, \quad dx = 2\cos\theta \, d\theta]$$

$$2\int (1-\cos 2\theta) \, d\theta = 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = 2(\theta - \sin\theta\cos\theta) + C = 2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + C.$$



□

(c) $\int \frac{5x^2-6x-5}{(x-1)^2(x+2)} \, dx$.

$$\frac{5x^2-6x-5}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2},$$

$$5x^2 - 6x - 5 = a(x-1)(x+2) + b(x+2) + c(x-1)^2.$$

Setting $x = 1$ gives $-6 = 3b$, so $b = -2$.
Setting $x = -2$ gives $27 = 9c$, so $c = 3$.
Therefore,

$$5x^2 - 6x - 5 = a(x-1)(x+2) - 2(x+2) + 3(x-1)^2.$$

Setting $x = 0$ gives $-5 = -2a - 4 + 3$, so $a = 2$.
Thus,

$$\int \frac{5x^2-6x-5}{(x-1)^2(x+2)} \, dx = \int \left(\frac{2}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x+2}\right) dx = 2\ln|x-1| + \frac{2}{x-1} + 3\ln|x+2| + C. \quad \square$$

(d) $\int (\sin 4x)^3(\cos 4x)^2\,dx$.

$$\int (\sin 4x)^3(\cos 4x)^2\,dx = \int (\sin 4x)^2(\cos 4x)^2(\sin 4x\,dx) = \int \left(1-(\cos 4x)^2\right)(\cos 4x)^2(\sin 4x\,dx) =$$

$$\left[u = \cos 4x, \quad du = -4\sin 4x\,dx, \quad dx = \frac{du}{-4\sin 4x}\right]$$

$$\int (1-u^2)u^2(\sin 4x)\left(\frac{du}{-4\sin 4x}\right) = \frac{1}{4}\int (u^4-u^2)\,du = \frac{1}{4}\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C =$$

$$\frac{1}{4}\left(\frac{1}{5}(\cos 4x)^5 - \frac{1}{3}(\cos 4x)^3\right) + C. \quad \square$$

(e) $\int (\sin 4x)^2(\cos 4x)^2\,dx$.

$$\int (\sin 4x)^2(\cos 4x)^2\,dx = \int \frac{1}{2}(1-\cos 8x)\cdot\frac{1}{2}(1+\cos 8x)\,dx = \frac{1}{4}\int \left(1-(\cos 8x)^2\right)\,dx = \frac{1}{4}\int (\sin 8x)^2\,dx =$$

$$\frac{1}{8}\int (1-\cos 16x)\,dx = \frac{1}{8}\left(x - \frac{1}{16}\sin 16x\right) + C. \quad \square$$

(f) $\int \frac{1}{(-3-4x-x^2)^{3/2}}\,dx$.

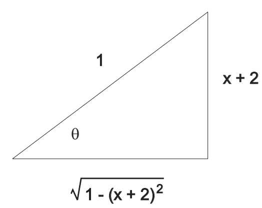
I need to complete the square. Note that $\frac{-4}{2} = -2$ and $(-2)^2 = 4$. Then

$$-3-4x-x^2 = -(x^2+4x+3) = -(x^2+4x+4-1) = -\left[(x+2)^2-1\right] = 1-(x+2)^2.$$

So

$$\int \frac{1}{(-3-4x-x^2)^{3/2}}\,dx = \int \frac{1}{(1-(x+2)^2)^{3/2}}\,dx = \int \frac{1}{(1-(\sin\theta)^2)^{3/2}}(\cos\theta\,d\theta) = \int \frac{1}{(\cos\theta)^3}(\cos\theta\,d\theta) =$$

$$[x+2 = \sin\theta, \quad dx = \cos\theta\,d\theta]$$



$$\int \frac{1}{(\cos\theta)^2}\,d\theta = \int (\sec\theta)^2\,d\theta = \tan\theta + C = \frac{x+2}{\sqrt{-3-4x-x^2}} + C. \quad \square$$

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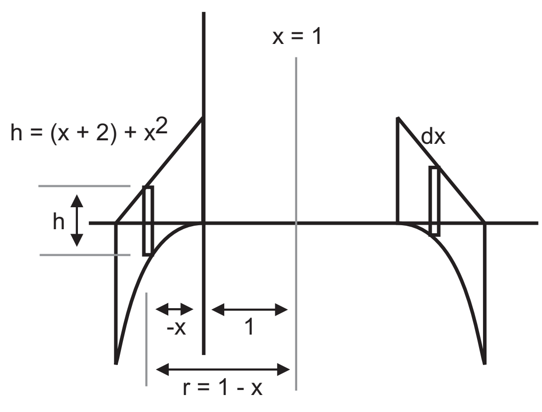
7. Let R be the region bounded above by $y = x+2$, bounded below by $y = -x^2$, and bounded on the sides by $x = -2$ and by the y-axis. Find the volume of the solid generated by revolving R about the line $x = 1$.



Most of the things in the picture are easy to understand — but why is $r = 1 - x$?

Notice that the *distance* from the y-axis to the side of the shell is $-x$, not x. Reason: x-values to the left of the y-axis are *negative*, but distances are always *positive*. Thus, I must use $-x$ to get a positive value for the distance.

As usual, r is the distance from the axis of revolution $x = 1$ to the side of the shell, which is $1 + (-x) = 1 - x$.

The left-hand cross-section extends from $x = -2$ to $x = 0$. You can check that if you plug x's between -2 and 0 into $r = 1 - x$, you get the correct distance from the side of the shell to the axis $x = 1$.

The volume is

$$V = \int_{-2}^{0} 2\pi(1-x)((x+2)+x^2)\,dx = 4\pi = \int_{-2}^{0} 2\pi\left(2 - x - x^3\right)\,dx = 2\pi\left[2x - \frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-2}^{0} =$$

$$20\pi \approx 62.83185. \quad \square$$

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8. Compute $\lim_{x\to\infty}\left(\sqrt{x^2+8x} - x\right)$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+8x} - x\right) = \lim_{x\to\infty}\left(\sqrt{x^2+8x} - x\right)\cdot\frac{\sqrt{x^2+8x}+x}{\sqrt{x^2+8x}+x} = \lim_{x\to\infty}\frac{x^2+8x-x^2}{\sqrt{x^2+8x}+x} =$$

$$\lim_{x\to\infty}\frac{8x}{\sqrt{x^2+8x}+x} = \lim_{x\to\infty}\frac{8x}{\sqrt{x^2+8x}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{8}{\sqrt{1+\frac{8}{x}}+1} = \frac{8}{1+1} = 4. \quad \square$$

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9. Compute $\lim_{x\to\infty}\left(1 + \frac{2}{x}\right)^{3x}$.

Let $y = \left(1 + \frac{2}{x}\right)^{3x}$, so

$$\ln y = \ln\left(1+\frac{2}{x}\right)^{3x} = 3x\ln\left(1+\frac{2}{x}\right).$$

Then

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} 3x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{3x}} = \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{2}{x}}\right)\left(-\frac{2}{x^2}\right)}{-\frac{1}{3x^2}} = 6\lim_{x\to\infty}\frac{1}{1+\frac{2}{x}} = 6.$$

Therefore,

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x} = \lim_{x\to\infty} y = e^{(\lim_{x\to\infty}\ln y)} = e^6. \quad \square$$

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10. If $x = t + e^t$ and $y = t + t^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+3t^2}{1+e^t}.$$

When $t = 1$, $\dfrac{dy}{dx} = \dfrac{4}{1+e}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{dt}{dx}\right)\left(\frac{d}{dt}\left(\frac{dy}{dx}\right)\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\frac{1+3t^2}{1+e^t}}{1+e^t} =$$

$$\frac{\frac{(1+e^t)(6t)-(1+3t^2)(e^t)}{(1+e^t)^2}}{1+e^t} = \frac{(1+e^t)(6t)-(1+3t^2)(e^t)}{(1+e^t)^3}.$$

When $t = 1$, $\dfrac{d^2y}{dx^2} = \dfrac{6+2e}{(1+e)^3}$. □

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11. (a) Find the Taylor expansion at $c = 1$ for e^{2x}.

$$e^{2x} = e^{2(x-1)+2} = e^2 e^{2(x-1)} = e^2\left(1 + 2(x-1) + \frac{2^2(x-1)^2}{2!} + \frac{2^3(x-1)^3}{3!} + \cdots\right). \quad \square$$

(b) Find the Taylor expansion at $c = 1$ for $\dfrac{1}{3+x}$. What is the interval of convergence?

$$\frac{1}{3+x} = \frac{1}{4+(x-1)} = \frac{1}{4}\cdot\frac{1}{1+\frac{x-1}{4}} = \frac{1}{4}\cdot\frac{1}{1-\left(-\frac{x-1}{4}\right)} =$$

$$\frac{1}{4}\left(1 - \frac{x-1}{4} + \left(\frac{x-1}{4}\right)^2 - \left(\frac{x-1}{4}\right)^3 + \cdots\right).$$

The series converges for $-1 < \dfrac{x-1}{4} < 1$, i.e. for $-3 < x < 5$. □

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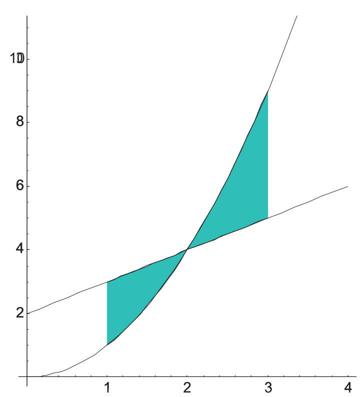
12. Find the area of the region which lies between the graphs of $y = x^2$ and $y = x+2$, from $x = 1$ to $x = 3$.



As the picture shows, the curves intersect. Find the intersection point:

$$x^2 = x+2, \quad x^2 - x - 2 = 0, \quad (x-2)(x+1) = 0, \quad x = 2 \quad \text{or} \quad x = -1.$$

On the interval $1 \le x \le 3$, the curves cross at $x = 2$. I'll use vertical rectangles. From $x = 1$ to $x = 2$, the top curve is $y = x + 2$ and the bottom curve is $y = x^2$. From $x = 2$ to $x = 3$, the top curve is $y = x^2$ and the bottom curve is $y = x + 2$. The area is

$$A = \int_1^2 \left((x+2) - x^2\right)\,dx + \int_2^3 \left(x^2 - (x+2)\right)\,dx = 3. \quad \square$$

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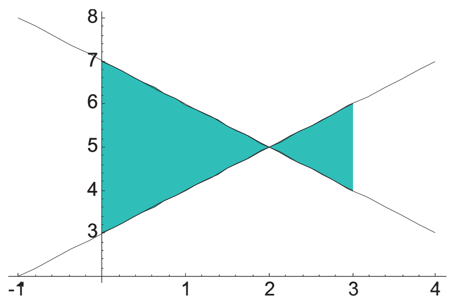
13. Find the area of the region between $y = x + 3$ and $y = 7 - x$ from $x = 0$ to $x = 3$.



As the picture shows, the curves intersect. Find the intersection point:

$$x + 3 = 7 - x, \quad 2x = 4, \quad x = 2.$$

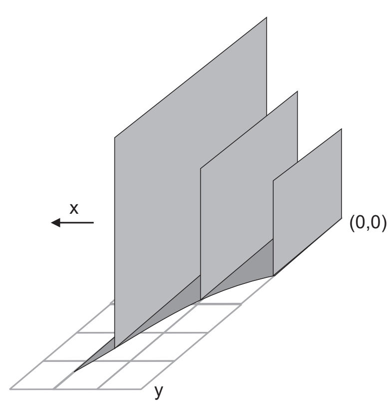
I'll use vertical rectangles. From $x = 0$ to $x = 2$, the top curve is $y = 7 - x$ and the bottom curve is $y = x + 3$. From $x = 2$ to $x = 3$, the top curve is $y = x + 3$ and the bottom curve is $y = 7 - x$. The area is

$$\int_0^2 ((7-x) - (x+3))\,dx + \int_2^3 ((x+3) - (7-x))\,dx = \int_0^2 (4 - 2x)\,dx + \int_2^3 (2x - 4)\,dx =$$

$$\left[4x - x^2\right]_0^2 + \left[x^2 - 4x\right]_2^3 = 4 + 1 = 5. \quad \square$$

---

14. The base of a solid is the region in the x-y-plane bounded above by the curve $y = e^x$, below by the x-axis, and on the sides by the lines $x = 0$ and $x = 1$. The cross-sections in planes perpendicular to the x-axis are squares with one side in the x-y-plane. Find the volume of the solid.



The volume is

$$V = \int_0^1 (e^x)^2\,dx = \int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1) \approx 3.19453. \quad \square$$

15. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(2^n)^3}$.

Apply the Ratio Test to the absolute value series:

$$\lim_{n\to\infty} \frac{\dfrac{|x-3|^{n+1}}{(n+1)(2^{n+1})^3}}{\dfrac{|x-3|^n}{n(2^n)^3}} = \lim_{n\to\infty} \frac{n}{n+1}\left(\frac{2^n}{2^{n+1}}\right)^3 \frac{|x-3|^{n+1}}{|x-3|^n} = \lim_{n\to\infty} \frac{n}{n+1}\cdot\frac{1}{8}|x-3| = \frac{1}{8}|x-3|.$$

The series converges for $\frac{1}{8}|x-3| < 1$, i.e. for $-5 < x < 11$.
At $x = 11$, the series is

$$\sum_{n=1}^{\infty} \frac{8^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

It's harmonic, so it diverges.
At $x = -5$, the series is

$$\sum_{n=1}^{\infty} \frac{(-8)^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

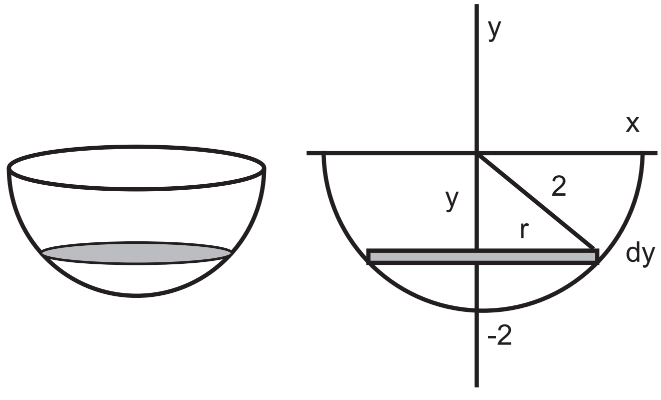
This is the alternating harmonic series, so it converges.
Therefore, the power series converges for $-5 \le x < 11$, and diverges elsewhere. □

16. Find the slope of the tangent line to the polar curve $r = \sin 2\theta$ at $\theta = \frac{\pi}{6}$.

When $\theta = \frac{\pi}{6}$, $r = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Since $\frac{dr}{d\theta} = 2\cos 2\theta$, when $\theta = \frac{\pi}{6}$, $\frac{dr}{d\theta} = 2\cos\frac{\pi}{3} = 1$.
The slope of the tangent line is

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}} = \frac{\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{2}\right)(1)}{\left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{\sqrt{3}}{2}\right)(1)} = \frac{5\sqrt{3}}{3} \approx 2.88675. \quad □$$

17. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.



I've drawn the tank in cross-section as a semicircle of radius 2 extending from $y = -2$ to $y = 0$.

Divide the volume of water up into circular slices. The radius of a slice is $r = \sqrt{4 - y^2}$, so the volume of a slice is $dV = \pi r^2\,dy = \pi(4 - y^2)\,dy$. The weight of a slice is $62.4\pi(4 - y^2)\,dy$, where I'm using 62.4 pounds per cubic foot as the density of water.

To pump a slice out of the top of the tank, it must be raised a distance of $-y$ feet. (The "-" is necessary to make y positive, since y is going from -2 to 0.)

The work done is

$$W = \int_{-2}^{0} 62.4\pi(-y)(4 - y^2)\,dy = 62.4\pi \int_{-2}^{0} (y^3 - 4y)\,dy = 62.4\pi \left[\frac{1}{4}y^4 - 2y^2\right]_{-2}^{0} =$$

$$249.6\pi \approx 784.14153 \text{ foot} - \text{pounds.} \quad \square$$

---

18. Let

$$x = \frac{\sqrt{3}}{2}t^2, \quad y = t - \frac{1}{4}t^3.$$

Find the length of the arc of the curve from $t = -2$ to $t = 2$.

$$\frac{dx}{dt} = \sqrt{3}t \quad \text{and} \quad \frac{dy}{dt} = 1 - \frac{3}{4}t^2,$$

so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 3t^2 + \left(1 - \frac{3}{4}t^2\right)^2 = 3t^2 + 1 - \frac{3}{2}t^2 + \frac{9}{16}t^4 = 1 + \frac{3}{2}t^2 + \frac{9}{16}t^4 = \left(1 + \frac{3}{4}t^2\right)^2.$$

Therefore,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1 + \frac{3}{4}t^2.$$

The length is

$$\int_{-2}^{2} \left(1 + \frac{3}{4}t^2\right) dt = \left[t + \frac{1}{4}t^3\right]_{-2}^{2} = 8. \quad \square$$

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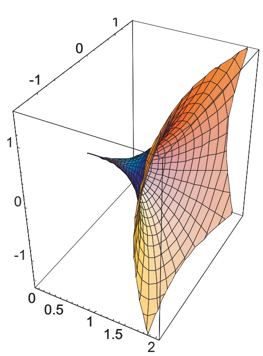
19. Find the area of the surface generated by revolving $y = \frac{1}{3}x^3$, $0 \le x \le 2$, about the x-axis.



The derivative is

$$\frac{dy}{dx} = x^2, \quad \text{so} \quad \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} = \sqrt{x^4 + 1}.$$

The curve is being revolved about the x-axis, so the radius of revolution is $R = y = \dfrac{1}{3}x^3$. The area of the surface is

$$S = \int_0^2 2\pi\left(\frac{1}{3}x^3\right)\sqrt{x^4+1}\,dx = \frac{2\pi}{3}\int_1^{17} u^{1/2}\cdot x^3\left(\frac{du}{4x^3}\right) = \frac{\pi}{6}\int_1^{17} u^{1/2}\,du = \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_1^{17} =$$

$$\left[u = x^4+1, \quad du = 4x^3\,dx, \quad dx = \frac{du}{4x^3}; \quad x=0, u=1, \quad x=2, u=17\right]$$

$$\frac{\pi}{9}\left(17^{3/2}-1\right) \approx 24.11794. \quad \Box$$

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20. (a) Convert $(x-3)^2 + (y+4)^2 = 25$ to polar and simplify.

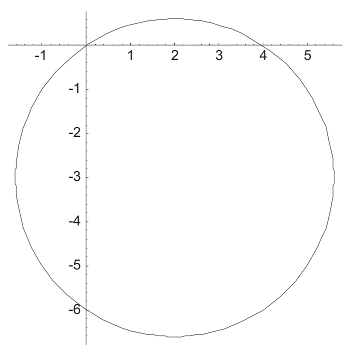
$$(x-3)^2 + (y+4)^2 = 25, \quad x^2 - 6x + 9 + y^2 + 8y + 16 = 25, \quad x^2 + y^2 = 6x - 8y,$$

$$r^2 = 6r\cos\theta - 8r\sin\theta, \quad r = 6\cos\theta - 8\sin\theta. \quad \Box$$

(b) Convert $r = 4\cos\theta - 6\sin\theta$ to rectangular and describe the graph.

$$r = 4\cos\theta - 6\sin\theta, \quad r^2 = 4r\cos\theta - 6r\sin\theta, \quad x^2 + y^2 = 4x - 6y, \quad x^2 - 4x + y^2 + 6y = 0,$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 13, \quad (x-2)^2 + (y+3)^2 = 13.$$



The graph is a circle of radius $\sqrt{13}$ centered at $(2, -3)$. $\Box$

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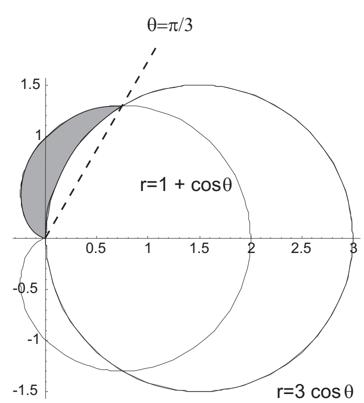
21. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.



Find the intersection points:

$$3\cos\theta = 1 + \cos\theta, \quad 2\cos\theta = 1, \quad \cos\theta = \frac{1}{2}, \quad \theta = \pm\frac{\pi}{3}.$$

I'll find the area of the shaded region and double it to get the total. The shaded area is

$$\left(\text{cardioid area from } \frac{\pi}{3} \text{ to } \pi\right) - \left(\text{circle area from } \frac{\pi}{3} \text{ to } \frac{\pi}{2}\right).$$

The cardioid area is

$$\int_{\pi/3}^{\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi} \left(1 + 2\cos\theta + (\cos\theta)^2\right)\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi} \left(1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right)\,d\theta =$$

$$\frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi} = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}.$$

The circle area is

$$\int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2\,d\theta = \frac{9}{4}\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\,d\theta = \frac{9}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} = \frac{3\pi}{8} - \frac{9}{16}\sqrt{3}.$$

Thus, the shaded area is

$$\left(\frac{\pi}{2} - \frac{9}{16}\sqrt{3}\right) - \left(\frac{3\pi}{8} - \frac{9}{16}\sqrt{3}\right) = \frac{\pi}{8}.$$

The total area is $2 \cdot \frac{\pi}{8} = \frac{\pi}{4} \approx 0.78540$. □

The best thing for being sad is to learn something. - Merlyn, in T. H. White's *The Once and Future King*