Math 161-04/05
11–28–2014

Review Problems for the Final

These problems are intended to help you study for the final. However, you shouldn't assume that each problem on this handout corresponds to a problem on the final. Nor should you assume that if a topic *doesn't* appear here, it *won't* appear on the final.

1. Compute the following derivatives:

(a) $\dfrac{d}{dx}\left(\dfrac{x}{(2x+1)^2}\right)$.

(b) $\dfrac{d}{dx}\left(\ln(e^x+1)+e^{\ln x+1}\right)$.

(c) $\dfrac{d}{dx}\ln\left[1+\ln\left(1+\ln x\right)\right]$.

(d) $\dfrac{d}{dx}7^{\sin x}$.

(e) $\dfrac{d}{dx}\sin\left(\dfrac{x^2+2}{x^2+3}\right)$.

(f) $\dfrac{d}{dx}\sqrt{\dfrac{\cos x+2}{\sin x+4}}$.

(g) $\dfrac{d}{dx}(x^5+4x^3-\ln x)(\sin x+37)$.

(h) $\dfrac{d}{dx}\sqrt{(x^3+5x+1)^7+x}$.

(i) $\dfrac{dy}{dx}$, where $x\sin y-\cos y+\cos 2y=0$.

(j) $\dfrac{dy}{dx}$, where $y=(e^x+2)^{x^2+3}$.

(k) y'' when $x=1$ and $y=1$, where $x^2y^3-x^3=x+y-2$.

2. Find the points on the curve $x^4+y^4=4xy$ where $y'=0$.

3. Compute the following derivatives:

(a) $\dfrac{d}{dx}\left((\arctan x)^5+\arctan(x^5)\right)$.

(b) $\dfrac{d}{dx}\arcsin(e^x+1)$.

(c) $\dfrac{d}{dx}(\tan^{-1}(\sec^{-1} x))$.

(d) $\dfrac{d}{dx}\dfrac{\cos x}{\cos^{-1} x}$.

(e) $\dfrac{d}{dx}\sqrt{\sin^{-1} x+x}$.

(f) $\dfrac{d}{dx}\tan^{-1}(e^x+1)$.

4. Prove that $e^x > 1 + x$ for $x \neq 0$.

5. Graph $y = 3x^{5/3} - \dfrac{3}{8}x^{8/3}$.

6. In the following problems, compute the limit, or show that it is undefined.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin 3x - \sin 5x}{\sin 2x}$.

(b) $\displaystyle\lim_{x\to\infty}\frac{4x+\sqrt{x^2+1}}{3x+7}$.

(c) $\displaystyle\lim_{h\to 0}\frac{(x+h)^{50}-x^{50}}{h}$.

(d) $\displaystyle\lim_{x\to 4}\frac{x^2-3x-4}{x^2-16}$.

(e) $\displaystyle\lim_{x\to 4^+}\sqrt{16-x^2}$.

(f) $\displaystyle\lim_{x\to 4}\frac{x^2-3x-4}{x^2-8x+16}$.

(g) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{\frac{1}{5}-\frac{1}{x+3}}$.

7. Compute the following integrals.

(a) $\displaystyle\int (e^{2x}+e^{3x})^2\,dx$.

(b) $\displaystyle\int \frac{2x+3}{(x+1)^4}\,dx$.

(c) $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$.

(d) $\displaystyle\int \frac{(3\ln x)^2+1}{x}\,dx$.

(e) $\displaystyle\int \frac{\cos x}{(\sin x)^2+2\sin x+1}\,dx$.

(f) $\displaystyle\int \frac{x^2+x}{\sqrt[3]{2-3x^2-2x^3}}\,dx$.

(g) $\displaystyle\int \frac{\left(\sec x^{1/3}\right)^2}{x^{2/3}}\,dx$.

(h) $\displaystyle\int_0^1 \frac{x\ln(x^2+1)}{x^2+1}\,dx$.

(i) $\displaystyle\int_1^2 \frac{f'(x)}{f(x)}\,dx$, if $f(1)=1$ and $f(2)=e$.

(j) $\int (x+1)4^{(x^2+2x+5)}\,dx.$

(l) $\int \frac{3^x}{2^x}\,dx.$

(m) $\int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx.$

(n) $\int \frac{1}{\sqrt{x}(1+x)}\,dx.$

8. Suppose $f(2) = 5$ and $f'(2) = -7$. Assuming that f has a differentiable inverse, what is $(f^{-1})'(5)$?

9. Find $(f^{-1})'(5)$ for $f(x) = x^7 + 8x - 4$.

10. The position of a bowl of potato salad at time t is

$$s(t) = 2t^3 - 30t^2.$$

(a) Find the velocity $v(t)$ and the acceleration $a(t)$.

(b) When is the velocity equal to 0? When is the acceleration equal to 0?

11. A population of flamingo lawn ornaments grows exponentially in Calvin's yard. There are 20 after 1 day and 60 after 4 days. How many are there after 6 days?

12. A bacon, sausage, onion, mushroom, and ham quiche is placed in a 400° oven. The initial temperature of the quiche is 80°; after 10 minutes, the quiche's temperature is 200°. What is the quiche's temperature 25 minutes after being placed in the oven?

13. A hot pastrami sandwich with a temperature of 150° is placed in a 70° room to cool. After 10 minutes, the temperature of the sandwich is 90°. When will the temperature be 80°?

14. Find the area of the region in the first quadrant bounded on the left by $y = x^2$, on the right by $x + y = 2$, and below by the x-axis.

15. Find the area of the region between $y = 12 - x^2$ and $y = x$ from $x = 0$ to $x = 4$.

16. Find the area of the region bounded by

$$x = y^2 - 2y \quad \text{and} \quad x = 4 - y^2.$$

17. Approximate the area under $y = (x - \sin x)^2$ from $x = 3$ to $x = 5$ using 20 rectangles of equal width, and using the midpoints of each subinterval to obtain the rectangles' heights.

18. Write the following sum using summation notation, then approximate its value to 5 decimal places:

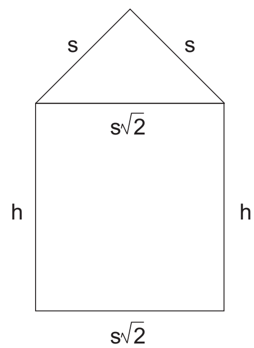
$$\frac{3+\sin 2}{1^2+1} + \frac{3+\sin 3}{2^2+2} + \frac{3+\sin 4}{3^2+3} + \cdots + \frac{3+\sin 41}{40^2+40}.$$

19. Calvin runs south toward Phoebe's house at 2 feet per second. Bonzo runs east away from Phoebe's house at 5 feet per second. At what rate is the distance between Calvin and Bonzo changing when Calvin is 50 feet from the house and Bonzo is 120 feet from the house?

20. A bird flies at a constant speed of 16 feet per second at a constant height of 48 feet. Its path takes it directly over a camera, which turns to track the bird. At what rate is the acute angle between the ground and the line of sight from the camera to the bird changing 4 second after it has passed above the camera?

21. Find the dimensions of the rectangle with the largest possible perimeter that can be inscribed in a semicircle of radius 1.

22. A window is made in the shape of a rectangle with an isosceles right triangle on top.



(a) Write down an expression for the *total area* of the window.

(b) Write down an expression for the *perimeter* of the window (that is, the length of the *outside* edge).

(c) If the perimeter is given to be 4, what value of s makes the total area a maximum?

23. A cylindrical can with a top and a bottom is to be made with 96π square inches of sheet metal with no waste. What values for the radius r and the height h give the can of largest volume?

24. (a) Find the absolute max and the absolute min of $y = x^3 - 12x + 5$ on the interval $0 \le x \le 5$.

(b) Find the absolute max and the absolute min of $f(x) = \frac{3}{4}x^{4/3} - 15x^{1/3}$ on the interval $-1 \le x \le 8$.

25. Use a limit of a rectangle sum to find the exact area under $y = x^2 + 3x$ from $x = 0$ to $x = 1$.

26. Given that $f(x) = x^5 + 4x^3 + 17$, what is $(f^{-1})'(22)$?

27. Find the largest interval containing $x = 1$ on which the function $f(x) = 3x^4 - 16x^3 + 1$ has an inverse f^{-1}.

28. (a) Compute $\frac{d}{dx}\int_4^{x^6} \sqrt{2+t^2}\,dt$.

(b) Compute $\int \left(\frac{d}{dx}\sqrt[3]{x^2+1}\right)\,dx$.

29. (a) Use the definition of the derivative as a limit to prove that $\frac{d}{dx}\frac{1}{x-4} = -\frac{1}{(x-4)^2}$.

*(b) Compute $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\sqrt{t^4+1}\,dt$.

30. Let

$$f(x) = \begin{cases} \dfrac{x+3}{6-x} & \text{if } x < 1 \\ 0.9 & \text{if } x = 1 \\ 3x^2 - 2.2 & \text{if } x > 1 \end{cases}.$$

Is f continuous at $x = 1$? Why or why not?

31. (a) Suppose that $f(3) = 5$ and $f'(x) = \dfrac{x^2}{x^2+16}$. Use differentials to approximate $f(2.99)$ to 5 places.

(b) A differentiable function satisfies $\dfrac{dy}{dx} = e^x \cos 3x$ and $y(0) = 0.1$. Use differentials to approximate $y(0.02)$.

32. Use 3 iterations of Newton's method starting at $x = 2$ to approximate a solution to $4 - x^2 = e^x$.

33. Prove that the function $f(x) = x^3 + 2x - \cos x + 5$ has exactly one root.

34. Compute the following limits:

(a) $\displaystyle\lim_{x\to 1} \frac{2 - 2e^{x-1}}{\sin(x-1)}$.

(b) $\displaystyle\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x - 3}{2x^3 - 6x^2 + x - 3}$.

(c) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$.

(d) $\displaystyle\lim_{x\to 0} \frac{\cos 2x}{x^2 + 3x + 1}$.

(e) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$.

(f) $\displaystyle\lim_{x\to +\infty} \left(\sqrt{x^2 + 4x} - x\right)$.

(g) $\displaystyle\lim_{x\to 0} \frac{\sin 4x + \tan 5x}{x\cos 3x + 12x}$.

(h) $\displaystyle\lim_{x\to 0} \frac{x - x\cos x}{x\sin x + 2x}$.

(i) $\displaystyle\lim_{x\to 0^+} \left(e^{2x} + x\right)^{1/x}$.

(j) $\displaystyle\lim_{x\to\infty} \frac{x^2 + \ln x}{5x^2 + x + 1}$.

(k) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{5}{x^2}\right)^{3x^2}$.

(l) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2 + 8x} - x\right)$.

(m) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x}$.

Solutions to the Review Problems for the Final

1. Compute the following derivatives.

(a)
$$\frac{d}{dx}\left(\frac{x}{(2x+1)^2}\right) = \frac{(2x+1)^2(1) - (x)(2)(2x+1)(2)}{(2x+1)^4}. \quad \square$$

(b) $$\frac{d}{dx}\left(\ln(e^x+1)+e^{\ln x+1}\right)=\frac{e^x}{e^x+1}+\left(\frac{1}{x}\right)\left(e^{\ln x+1}\right). \quad \square$$

(c) $$\frac{d}{dx}\ln\left[1+\ln\left(1+\ln x\right)\right]=\left(\frac{1}{1+\ln\left(1+\ln x\right)}\right)\left(\frac{1}{1+\ln x}\right)\left(\frac{1}{x}\right). \quad \square$$

(d) $$\frac{d}{dx}7^{\sin x}=(\ln 7)(7^{\sin x})(\cos x). \quad \square$$

(e) $$\frac{d}{dx}\sin\left(\frac{x^2+2}{x^2+3}\right)=\left(\cos\left(\frac{x^2+2}{x^2+3}\right)\right)\left(\frac{(x^2+3)(2x)-(x^2+2)(2x)}{(x^2+3)^2}\right). \quad \square$$

(f) $$\frac{d}{dx}\sqrt{\frac{\cos x+2}{\sin x+4}}=\frac{1}{2}\left(\frac{\cos x+2}{\sin x+4}\right)^{-1/2}\left(\frac{(\sin x+4)(-\sin x)-(\cos x+2)(\cos x)}{(\sin x+4)^2}\right). \quad \square$$

((g) $$\frac{d}{dx}(x^5+4x^3-\ln x)(\sin x+37)=(x^5+4x^3-\ln x)(\cos x)+(\sin x+37)\left(5x^4+12x^2-\frac{1}{x}\right). \quad \square$$

(h) $$\frac{d}{dx}\sqrt{(x^3+5x+1)^7+x}=\frac{1}{2}\left((x^3+5x+1)^7+x\right)^{-1/2}\left(7(x^3+5x+1)^6(3x^2+5)+1\right). \quad \square$$

(i) $\dfrac{dy}{dx}$, where $x\sin y-\cos y+\cos 2y=0$.

Differentiate implicitly, then solve for y':

$$\sin y+xy'\cos y+y'\sin y-2y'\sin 2y=0, \quad y'=\frac{\sin y}{2\sin 2y-x\cos y-\sin y}. \quad \square$$

(j) $\dfrac{dy}{dx}$, where $y = (e^x + 2)^{x^2+3}$.

Use logarithmic differentiation:

$$\ln y = \ln(e^x + 2)^{x^2+3} = (x^2 + 3)\ln(e^x + 2),$$

$$\frac{y'}{y} = 2x\ln(e^x + 2) + \frac{e^x(x^2 + 3)}{e^x + 2},$$

$$y' = (e^x + 2)^{x^2+3}\left(2x\ln(e^x + 2) + \frac{e^x(x^2 + 3)}{e^x + 2}\right). \quad \square$$

(k) y'' when $x = 1$ and $y = 1$, where $x^2y^3 - x^3 = x + y - 2$.

Differentiate implicitly:

$$3x^2y^2y' + 2xy^3 - 3x^2 = 1 + y'. \qquad (*)$$

Plug in $x = 1$ and $y = 1$:

$$3y' + 2 - 3 = 1 + y', \quad 2y' = 2, \quad y' = 1.$$

Now differentiate (*) implicitly:

$$3x^2y^2y'' + 6x^2y(y')^2 + 6xy^2y' + 6xy^2y' + 2y^3 - 6x = y''.$$

Note that the term $3x^2y^2y'$ produces *three* terms when the Product Rule is applied. Now set $x = 1$, $y = 1$, and $y' = 1$:

$$3y'' + 6 + 6 + 6 + 2 - 6 = y'', \quad 2y'' + 14 = 0, \quad y'' = -7. \quad \square$$

---

2. Find the points on the curve $x^4 + y^4 = 4xy$ where $y' = 0$.

Differentiate implicitly:

$$4x^3 + 4y^3y' = 4(xy' + y).$$

Set $y' = 0$:

$$\begin{aligned} 4x^3 + 0 &= 4(0 + y) \\ x^3 &= y \end{aligned}$$

Substitute $y = x^3$ into $x^4 + y^4 = 4xy$ and solve for x:

$$\begin{aligned} x^4 + (x^3)^4 &= 4x \cdot x^3 \\ x^4 + x^{12} &= 4x^4 \\ x^{12} - 3x^4 &= 0 \\ x^4(x^8 - 3) &= 0 \end{aligned}$$

$x^4 = 0$ gives $x = 0$, so $y = 0^3 = 0$. The point is $(0, 0)$.

$x^8 - 3 = 0$ gives $x^8 = 3$, or $x = \pm 3^{1/8}$. First, $x = 3^{1/8}$ gives $y = (3^{1/8})^3 = 3^{3/8}$. Second, $x = -3^{1/8}$ gives $y = (-3^{1/8})^3 = -3^{3/8}$. The points are $(3^{1/8}, 3^{3/8})$ and $(-3^{1/8}, -3^{3/8})$.

The points are $(0, 0)$, $(3^{1/8}, 3^{3/8})$, and $(-3^{1/8}, -3^{3/8})$. $\square$

---

3. Compute the following derivatives:

(a) $\dfrac{d}{dx}\left((\arctan x)^5 + \arctan(x^5)\right)$.

$$\frac{d}{dx}\left((\arctan x)^5 + \arctan(x^5)\right) = 5(\arctan x)^4 \cdot \frac{1}{x^2+1} + \frac{5x^4}{1+x^{10}}. \quad \square$$

(b) $\dfrac{d}{dx}\arcsin(e^x + 1)$.

$$\frac{d}{dx}\arcsin(e^x+1) = \frac{e^x}{\sqrt{1-(e^x+1)^2}}. \quad \square$$

(c) $\dfrac{d}{dx}(\tan^{-1}(\sec^{-1} x))$.

$$\frac{d}{dx}(\tan^{-1}(\sec^{-1} x)) = \left(\frac{1}{1+(\sec^{-1} x)^2}\right)\left(\frac{1}{|x|\sqrt{x^2-1}}\right). \quad \square$$

(d) $\dfrac{d}{dx}\dfrac{\cos x}{\cos^{-1} x}$.

$$\frac{d}{dx}\frac{\cos x}{\cos^{-1} x} = \frac{(\cos^{-1} x)(-\sin x) - (\cos x)\left(\dfrac{-1}{\sqrt{1-x^2}}\right)}{(\cos^{-1} x)^2}. \quad \square$$

(e) $\dfrac{d}{dx}\sqrt{\sin^{-1} x + x}$.

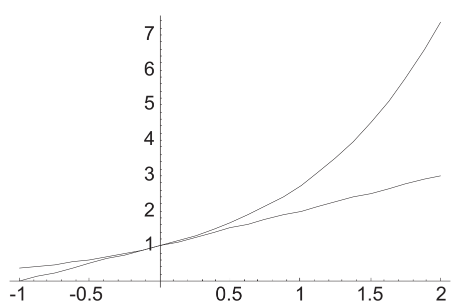
$$\frac{d}{dx}\sqrt{\sin^{-1} x + x} = \frac{1}{2}\left(\sin^{-1} x + x\right)^{-1/2}\left(\frac{1}{\sqrt{1-x^2}} + 1\right). \quad \square$$

(f) $\dfrac{d}{dx}\tan^{-1}(e^x + 1)$.

$$\frac{d}{dx}\tan^{-1}(e^x+1) = \frac{e^x}{1+(e^x+1)^2}. \quad \square$$

---

4. Prove that $e^x > 1 + x$ for $x \neq 0$.



The result is true, as the picture shows. However, a picture is not a proof.

Let $f(x) = e^x - (1+x)$. f measures the distance between the two curves. The derivative is $f'(x) = e^x - 1$. $f' = 0$ at $x = 0$; since $f''(x) = e^x$ and $f''(0) = 1 > 0$, the critical point is a local min. Since it is the only critical point, it is an absolute min.

Thus, the minimum vertical distance between the curves occurs at $x = 0$, when it is $f(0) = 0$. Since this is an absolute min, it follows that $f(x) > 0$ for $x \neq 0$ — that is, $e^x - (1+x) > 0$ for $x \neq 0$. This is equivalent to what I wanted to show. $\square$

5. Graph $y = 3x^{5/3} - \frac{3}{8}x^{8/3}$.

The function is defined for all x (you can take the cube root of any number).

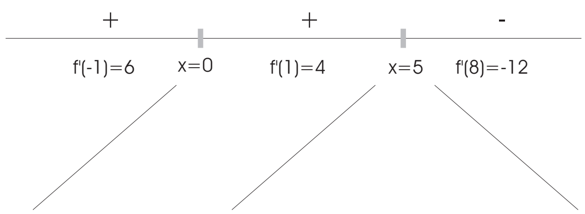
Since $y = 3x^{5/3}\left(1 - \frac{1}{8}x\right)$, the x-intercepts are $x = 0$ and $x = 8$. The y-intercept is $y = 0$.

The derivatives are

$$y' = 5x^{2/3} - x^{5/3} = x^{2/3}(5 - x), \quad y'' = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{5}{3} \cdot \frac{2 - x}{x^{1/3}}.$$

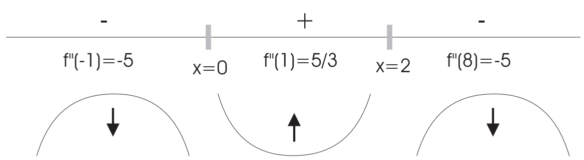
(In working with these kinds of expressions, it is better to have a sum of terms when you're differentiating. On the other hand, it is better to have everything together in factored form when you set up the sign charts. Notice how I used these two forms in the derivatives above.)

$y' = 0$ for $x = 0$ and at $x = 5$. y' is defined for all x.



The function increases for $x \le 5$ and decreases for $x \ge 5$. There is a local max at $x = 5$, $y \approx 16.44760$.

$y'' = 0$ at $x = 2$. y'' is undefined at $x = 0$.

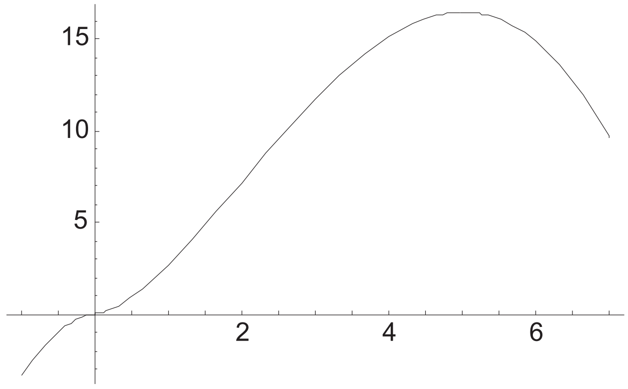


The graph is concave down for $x < 0$ and for $x > 2$. The graph is concave up for $0 < x < 2$. $x = 0$ and $x = 2$ are inflection points.

Since the function is defined for all x, there are no vertical asymptotes.

$$\lim_{x \to +\infty}\left(3x^{5/3} - \frac{3}{8}x^{8/3}\right) = -\infty \quad \text{and} \quad \lim_{x \to -\infty}\left(3x^{5/3} - \frac{3}{8}x^{8/3}\right) = -\infty.$$

The graph goes downward on the far left and far right.



6. In the following problems, compute the limit, or show that it is undefined.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x - \sin 5x}{\sin 2x}$.

$$\lim_{x\to 0} \frac{\sin 3x - \sin 5x}{\sin 2x} = \lim_{x\to 0} \frac{\dfrac{\sin 3x}{x} - \dfrac{\sin 5x}{x}}{\dfrac{\sin 2x}{x}} = \lim_{x\to 0} \frac{3 \cdot \dfrac{\sin 3x}{3x} - 5 \cdot \dfrac{\sin 5x}{5x}}{2 \cdot \dfrac{\sin 2x}{2x}} = \frac{3-5}{2} = -1. \quad \square$$

(b) $\displaystyle\lim_{x\to\infty} \frac{4x + \sqrt{x^2+1}}{3x+7}$.

$$\lim_{x\to\infty} \frac{4x + \sqrt{x^2+1}}{3x+7} = \lim_{x\to\infty} \frac{4 + \sqrt{1 + \dfrac{1}{x^2}}}{3 + \dfrac{7}{x}} = \frac{4+1}{3+0} = \frac{5}{3}. \quad \square$$

(c) $\displaystyle\lim_{h\to 0} \frac{(x+h)^{50} - x^{50}}{h}$.

$$\lim_{h\to 0} \frac{(x+h)^{50} - x^{50}}{h} = f'(x),$$

for $f(x) = x^{50}$. Hence, by the Power Rule,

$$\lim_{h\to 0} \frac{(x+h)^{50} - x^{50}}{h} = 50x^{49}. \quad \square$$

(d) $\displaystyle\lim_{x\to 4} \frac{x^2 - 3x - 4}{x^2 - 16}$.

$$\lim_{x\to 4} \frac{x^2 - 3x - 4}{x^2 - 16} = \lim_{x\to 4} \frac{(x+1)(x-4)}{(x-4)(x+4)} = \lim_{x\to 4} \frac{x+1}{x+4} = \frac{5}{8}. \quad \square$$

(e) $\displaystyle\lim_{x\to 4^+} \sqrt{16 - x^2}$.

For x close to 4 but larger than 4 — e.g. $x = 4.01$ — $16 - x^2$ is negative. Since the square root of a negative number is undefined, the limit is undefined. $\square$

(f) $\displaystyle\lim_{x\to 4} \frac{x^2 - 3x - 4}{x^2 - 8x + 16}$.

$$\lim_{x\to 4} \frac{x^2 - 3x - 4}{x^2 - 8x + 16} = \lim_{x\to 4} \frac{(x-4)(x+1)}{(x-4)^2} = \lim_{x\to 4} \frac{x+1}{x-4}.$$

Plugging in gives $\dfrac{5}{0}$. Moreover,

$$\lim_{x\to 4^+} \frac{x+1}{x-4} = +\infty \quad \text{and} \quad \lim_{x\to 4^-} \frac{x+1}{x-4} = -\infty.$$

Hence, the limit is undefined. ☐

(g) $\lim_{x\to 2} \dfrac{x^2-4}{\dfrac{1}{5}-\dfrac{1}{x+3}}$.

$$\lim_{x\to 2} \frac{x^2-4}{\frac{1}{5}-\frac{1}{x+3}} = \lim_{x\to 2} \frac{x^2-4}{\frac{x+3}{5(x+3)}-\frac{5}{5(x+3)}} = \lim_{x\to 2} \frac{x^2-4}{\frac{x+3-5}{5(x+3)}} = \lim_{x\to 2} \frac{x^2-4}{\frac{x-2}{5(x+3)}} = \lim_{x\to 2} \frac{(x-2)(x+2)}{\frac{x-2}{5(x+3)}} =$$

$$\lim_{x\to 2}(x-2)(x+2)\cdot\frac{5(x+3)}{x-2} = \lim_{x\to 2} 5(x+3)(x+2) = 100. \quad \square$$

---

7. Compute the following integrals.

(a) $\int (e^{2x}+e^{3x})^2\,dx$.

$$\int (e^{2x}+e^{3x})^2\,dx = \int \left(e^{4x}+2e^{5x}+e^{6x}\right)\,dx = \frac{1}{4}e^{4x}+\frac{2}{5}e^{5x}+\frac{1}{6}e^{6x}+C. \quad \square$$

(b) $\int \dfrac{2x+3}{(x+1)^4}\,dx$.

$$\int \frac{2x+3}{(x+1)^4}\,dx = \int \frac{2(u-1)+3}{u^4}\,du = \int \frac{2u+1}{u^4}\,du = \int \left(\frac{2}{u^3}+\frac{1}{u^4}\right)\,du =$$

$$[u=x+1, \quad du=dx, \quad x=u-1]$$

$$-\frac{1}{u^2}-\frac{1}{3u^3}+C = -\frac{1}{(x+1)^2}-\frac{1}{3(x+1)^3}+C. \quad \square$$

(c) $\int \dfrac{e^{2x}}{e^{2x}+1}\,dx$.

$$\int \frac{e^{2x}}{e^{2x}+1}\,dx = \int \frac{e^{2x}}{u}\cdot\frac{du}{2e^{2x}} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|e^{2x}+1|+C.$$

$$\left[u=e^{2x}+1, \quad du=2e^{2x}\,dx, \quad dx=\frac{du}{2e^{2x}}\right] \quad \square$$

(d) $\int \dfrac{(3\ln x)^2+1}{x}\,dx$.

$$\int \frac{(3\ln x)^2+1}{x}\,dx = \int \frac{3u^2+1}{x}\cdot x\,du = \int (3u^2+1)\,du = u^3+u+C = (\ln x)^3+\ln x+C. \quad \square$$

$$\left[u=\ln x, \quad du=\frac{dx}{x}, \quad dx=x\,du\right]$$

(e) $\displaystyle\int \frac{\cos x}{(\sin x)^2 + 2\sin x + 1}\,dx.$

$$\int \frac{\cos x}{(\sin x)^2 + 2\sin x + 1}\,dx = \int \frac{\cos x}{u^2 + 2u + 1} \cdot \frac{du}{\cos x} = \int \frac{du}{u^2 + 2u + 1} =$$

$$\left[u = \sin x, \quad du = \cos x\,dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int \frac{du}{(u+1)^2} = -\frac{1}{u+1} + C = -\frac{1}{\sin x + 1} + C. \quad \square$$

(f) $\displaystyle\int \frac{x^2 + x}{\sqrt[3]{2 - 3x^2 - 2x^3}}\,dx.$

$$\int \frac{x^2 + x}{\sqrt[3]{2 - 3x^2 - 2x^3}}\,dx = \int \frac{x^2 + x}{\sqrt[3]{u}} \cdot \frac{du}{-6(x^2 + x)} = -\frac{1}{6}\int u^{-1/3}\,du =$$

$$\left[u = 2 - 3x^2 - 2x^3, \quad du = (-6x - 6x^2)\,dx, \quad dx = \frac{du}{-6(x^2 + x)}\right]$$

$$-\frac{1}{6} \cdot \frac{3}{2}u^{2/3} + C = -\frac{1}{4}(2 - 3x^2 - 2x^3)^{2/3} + C. \quad \square$$

(g) $\displaystyle\int \frac{\left(\sec x^{1/3}\right)^2}{x^{2/3}}\,dx.$

$$\int \frac{\left(\sec x^{1/3}\right)^2}{x^{2/3}}\,dx = \int \frac{(\sec u)^2}{x^{2/3}} \cdot 3x^{2/3}\,du = 3\int (\sec u)^2\,du = 3\tan u + C = 3\tan x^{1/3} + C.$$

$$\left[u = x^{1/3}, \quad du = \frac{1}{3}x^{-2/3}\,dx, \quad dx = 3x^{2/3}\,du\right] \quad \square$$

(h) $\displaystyle\int_0^1 \frac{x\ln(x^2 + 1)}{x^2 + 1}\,dx.$

$$\int_0^1 \frac{x\ln(x^2 + 1)}{x^2 + 1}\,dx = \int_1^2 \frac{x\ln u}{u} \cdot \frac{du}{2x} =$$

$$\left[u = x^2 + 1, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}; \quad x = 0, u = 1; \quad x = 1, u = 2\right]$$

$$\frac{1}{2}\int_1^2 \frac{\ln u}{u}\,du = \frac{1}{2}\int_0^{\ln 2} \frac{w}{u} \cdot u\,dw = \frac{1}{2}\int_0^{\ln 2} w\,dw = \frac{1}{2}\left[\frac{1}{2}w^2\right]_0^{\ln 2} = \frac{1}{4}(\ln 2)^2 \approx 0.12011.$$

$$\left[w = \ln u, \quad dw = \frac{du}{u}, \quad du = u\,dw; \quad u = 1, w = 0; \quad u = 2, w = \ln 2\right] \quad \square$$

(i) $\displaystyle\int_1^2 \frac{f'(x)}{f(x)}\,dx$, if $f(1) = 1$ and $f(2) = e$.

$$\int_1^2 \frac{f'(x)}{f(x)}\,dx = \int_1^e \frac{f'(x)}{u} \cdot \frac{du}{f'(x)} = \int_1^e \frac{du}{u} = [\ln|u|]_1^e = 1.$$

$$\left[u = f(x), \quad du = f'(x)\,dx, \quad dx = \frac{du}{f'(x)}; \quad x = 1, u = f(1) = 1; \quad x = 2, u = f(2) = e\right] \quad \square$$

(j) $\displaystyle\int (x+1)4^{(x^2+2x+5)}\,dx$.

$$\int (x+1)4^{(x^2+2x+5)}\,dx = \int (x+1)4^u \cdot \frac{du}{2(x+1)} = \frac{1}{2}\int 4^u\,du = \frac{1}{2}\cdot\frac{1}{\ln 4}4^u + C = \frac{1}{2\ln 4}4^{(x^2+2x+5)} + C.$$

$$\left[u = x^2+2x+5, \quad du = (2x+2)\,dx = 2(x+1)\,dx, \quad dx = \frac{du}{2(x+1)}\right] \quad \square$$

(l) $\displaystyle\int \frac{3^x}{2^x}\,dx$.

$$\int \frac{3^x}{2^x}\,dx = \int \left(\frac{3}{2}\right)^x dx = \int 1.5^x\,dx = \frac{1}{\ln 1.5}1.5^x + c. \quad \square$$

(m) $\displaystyle\int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx$.

$$\int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx = \int \frac{1}{e^x\sqrt{1-u^2}}\cdot(-e^x\,du) = -\int \frac{1}{\sqrt{1-u^2}}\,du = -\sin^{-1}u + C = -\sin^{-1}e^{-x} + C.$$

$$\left[u = e^{-x}, \quad du = -e^{-x}\,dx, \quad dx = -e^x\,du\right] \quad \square$$

(n) $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$.

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{1}{\sqrt{x}(1+u^2)}\cdot 2\sqrt{x}\,du = 2\int \frac{1}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C.$$

$$\left[u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\,dx, \quad dx = 2\sqrt{x}\,du\right] \quad \square$$

---

8. Suppose $f(2) = 5$ and $f'(2) = -7$. Assuming that f has a differentiable inverse, what is $(f^{-1})'(5)$?

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)} = -\frac{1}{7}. \quad \square$$

---

9. Find $(f^{-1})'(5)$ for $f(x) = x^7 + 8x - 4$.

First, note that $f'(x) = 7x^6 + 8$.
Also, $f(1) = 1 + 8 - 4 = 5$, so $f^{-1}(5) = 1$.
Hence,

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(1)} = \frac{1}{7+8} = \frac{1}{15}. \quad \square$$

---

10. The position of a bowl of potato salad at time t is

$$s(t) = 2t^3 - 30t^2.$$

(a) Find the velocity $v(t)$ and the acceleration $a(t)$.

$$v(t) = s'(t) = 6t^2 - 60t = 6t(t-10), \quad a(t) = v'(t) = 12t - 60 = 12(t-5). \quad \square$$

(b) When is the velocity equal to 0? When is the acceleration equal to 0?

The velocity is 0 at $t = 0$ and at $t = 10$. The acceleration is 0 at $t = 5$. $\square$

---

11. A population of flamingo lawn ornaments grows exponentially in Calvin's yard. There are 20 after 1 day and 60 after 4 days. How many are there after 6 days?

Let F be the number of flamingos after t days. Then

$$F = F_0 e^{kt}.$$

When $t = 1$, $F = 20$:

$$20 = F_0 e^k.$$

When $t = 4$, $F = 60$:

$$60 = F_0 e^{4k}.$$

Divide $60 = F_0 e^{4k}$ by $20 = F_0 e^k$ and solve for k:

$$\frac{60}{20} = \frac{F_0 e^{4k}}{F_0 e^k}, \quad 3 = e^{3k}, \quad \ln 3 = \ln e^{3k} = 3k, \quad k = \frac{\ln 3}{3}.$$

Plug this back into $20 = F_0 e^k$:

$$20 = F_0 e^{(\ln 3)/3}, \quad F_0 = \frac{20}{e^{(\ln 3)/3}}.$$

Hence,

$$F = \frac{20}{e^{(\ln 3)/3}} e^{(t \ln 3)/3}.$$

When $t = 6$,

$$F = \frac{20}{e^{(\ln 3)/3}} e^{2 \ln 3} \approx 124.80503 \text{ flamingos} \quad \square$$

---

12. A bacon, sausage, onion, mushroom, and ham quiche is placed in a 400° oven. The initial temperature of the quiche is 80°; after 10 minutes, the quiche's temperature is 200°. What is the quiche's temperature 25 minutes after being placed in the oven?

The temperature of the oven is $T_e = 400$ and the initial temperature is $T_0 = 80$. So

$$T = 400 + (80 - 400)e^{kt} = 400 - 320e^{kt}.$$

When $t = 10$, the temperature is $T = 200$:

$$\begin{aligned} 200 &= 400 - 320e^{10k} \\ -200 &= -320e^{10k} \\ \frac{5}{8} &= e^{10k} \\ \ln\frac{5}{8} &= 10k \\ \frac{1}{10}\ln\frac{5}{8} &= k \end{aligned}$$

Thus,

$$T = 400 - 320\exp\left(t \cdot \frac{1}{10}\ln\frac{5}{8}\right).$$

Remember that "exp(FOO)" is another way of writing "e^{FOO}". Also, notice that I wrote the "t" to the left of the value for k, because I don't mean t to multiply the "$\frac{5}{8}$".

Set $t = 25$:

$$T = 400 - 320\exp\left(25 \cdot \frac{1}{10}\ln\frac{5}{8}\right) = 301.17882\ldots. \quad \square$$

---

13. A hot pastrami sandwich with a temperature of 150° is placed in a 70° room to cool. After 10 minutes, the temperature of the sandwich is 90°. When will the temperature be 80°?

Let T be the temperature at time t. The initial temperature is $T_0 = 150$, and the room's temperature is 70, so

$$T = 70 + (150 - 70)e^{kt} \quad \text{or} \quad T = 70 + 80e^{kt}.$$

When $t = 10$, $T = 90$:

$$90 = 70 + 80e^{10k}, \quad 20 = 80e^{10k}, \quad \frac{1}{4} = e^{10k}, \quad \ln\frac{1}{4} = \ln e^{10k},$$

$$\ln\frac{1}{4} = 10k, \quad k = \frac{1}{10}\ln\frac{1}{4}.$$

Hence,

$$T = 70 + 80e^{t(1/10)\ln(1/4)}.$$

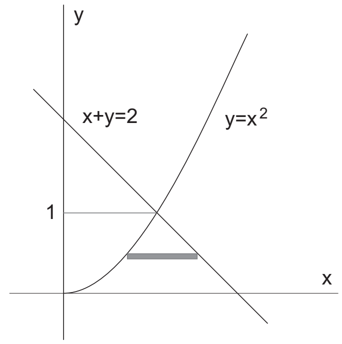
Set $T = 80$ and solve for t:

$$80 = 70 + 80e^{t(1/10)\ln(1/4)}, \quad 10 = 80e^{t(1/10)\ln(1/4)}, \quad \frac{1}{8} = e^{t(1/10)\ln(1/4)}, \quad \ln\frac{1}{8} = \ln e^{t(1/10)\ln(1/4)},$$

$$\ln\frac{1}{8} = t \cdot \frac{1}{10}\ln\frac{1}{4}, \quad t = \frac{\ln\frac{1}{8}}{\frac{1}{10}\ln\frac{1}{4}} \approx 15 \text{ minutes.} \quad \square$$

---

14. Find the area of the region in the first quadrant bounded on the left by $y = x^2$, on the right by $x + y = 2$, and below by the x-axis.



The curves intersect at the point $x = 1$, $y = 1$. (You can see this by solving $y = x^2$ and $x + y = 2$ simultaneously.)

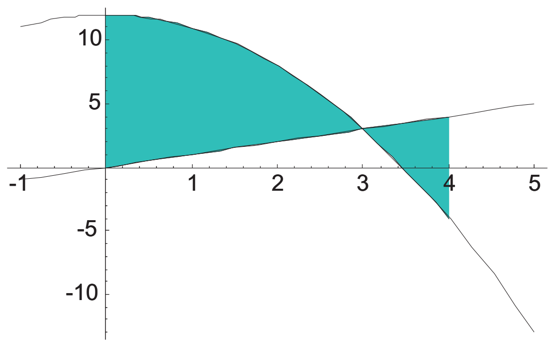
Divide the region up into horizontal rectangles. A typical rectangle has width dy. The *right* end of a rectangle is on $x = 2 - y$; the *left* end of a rectangle is on $x = \sqrt{y}$ (i.e. $y = x^2$). Therefore, the length of a rectangle is $2 - y - \sqrt{y}$, and the area of a rectangle is $(2 - y - \sqrt{y})\, dy$.

The area is

$$A = \int_0^1 (2 - y - \sqrt{y})\, dy = \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]_0^1 = \frac{5}{6} \approx 0.83333. \quad \square$$

---

15. Find the area of the region between $y = 12 - x^2$ and $y = x$ from $x = 0$ to $x = 4$.



Find the intersection point:

$$12 - x^2 = x, \quad x^2 + x - 12 = 0, \quad (x + 4)(x - 3) = 0, \quad x = -4 \quad \text{or} \quad x = 3.$$

$x = 3$ is the intersection point between 0 and 4.

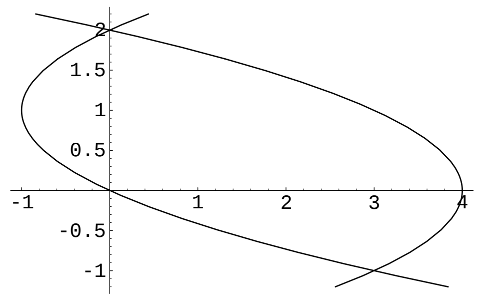
Use vertical rectangles. Between $x = 0$ and $x = 3$, the top curve is $y = 12 - x^2$ and the bottom curve is $y = x$. Between $x = 3$ and $x = 4$, the top curve is $y = x$ and the bottom curve is $y = 12 - x^2$. The area is

$$\int_0^3 (12 - x^2 - x)\, dx + \int_3^4 (x - (12 - x^2))\, dx = \left[12x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^3 + \left[\frac{1}{2}x^2 - 12x + \frac{1}{3}x^3\right]_3^4 = \frac{79}{3} \approx 26.33333. \quad \square$$

---

16. Find the area of the region bounded by

$$x = y^2 - 2y \quad \text{and} \quad x = 4 - y^2.$$



Find the intersection points:

$$\begin{aligned} y^2 - 2y &= 4 - y^2 \\ 2y^2 - 2y - 4 &= 0 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \end{aligned}$$

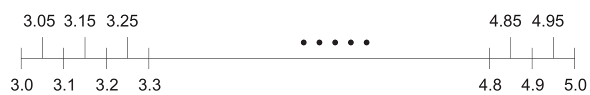
The curves intersect at $y = -1$ and at $y = 2$. The left-hand curve is $x = y^2 - 2y$ and the right-hand curve is $x = 4 - y^2$. Using horizontal rectangles, the area is

$$\int_{-1}^{2} [(4 - y^2) - (y^2 - 2y)]\, dy = \int_{-1}^{2} (4 + 2y - 2y^2)\, dy = \left[4y + y^2 - \frac{2}{3}y^3\right]_{-1}^{2} = 9. \quad \square$$

---

17. Approximate the area under $y = (x - \sin x)^2$ from $x = 3$ to $x = 5$ using 20 rectangles of equal width, and using the midpoints of each subinterval to obtain the rectangles' heights.

The width of each rectangle is $\Delta x = \dfrac{5-3}{20} = 0.1$.



The midpoints start at 3.05 and go to 4.95 in steps of size 0.1.
The calculator command to compute the sum is:

sum(seq((x - sin(x))∧2, x, 3.05, 4.95, 0.1)) * 0.1

The answer is $44.71066\ldots$. $\square$

---

18. Write the following sum using summation notation, then approximate its value to 5 decimal places:

$$\frac{3 + \sin 2}{1^2 + 1} + \frac{3 + \sin 3}{2^2 + 2} + \frac{3 + \sin 4}{3^2 + 3} + \cdots + \frac{3 + \sin 41}{40^2 + 40}.$$

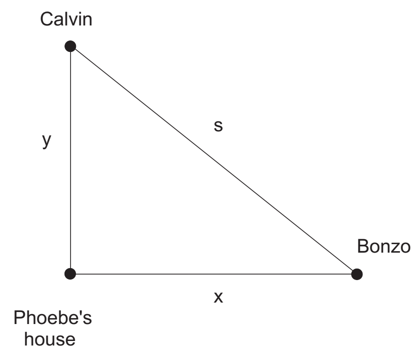
$$\frac{3 + \sin 2}{1^2 + 1} + \frac{3 + \sin 3}{2^2 + 2} + \frac{3 + \sin 4}{3^2 + 3} + \cdots + \frac{3 + \sin 41}{40^2 + 40} = \sum_{n=1}^{40} \frac{3 + \sin(n+1)}{n^2 + n}.$$

The calculator command to do the sum is:

sum(seq((3 + sin(x + 1))/(x∧2 + x), x, 1, 40))
The answer is $3.31192\ldots$. $\square$

19. Calvin runs south toward Phoebe's house at 2 feet per second. Bonzo runs east away from Phoebe's house at 5 feet per second. At what rate is the distance between Calvin and Bonzo changing when Calvin is 50 feet from the house and Bonzo is 120 feet from the house?

Let x be the distance from Bonzo to the house, let y be the distance from Calvin to the house, and let s be the distance between Calvin and Bonzo.



By Pythagoras,

$$s^2 = x^2 + y^2.$$

Differentiate with respect to t:

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}, \quad s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

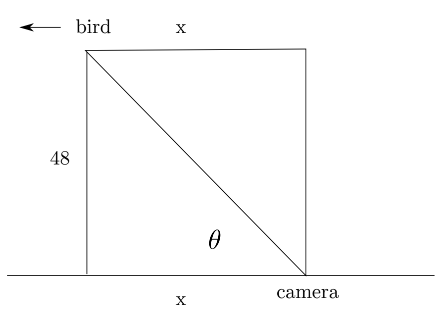
$\dfrac{dx}{dt} = 5$ and $\dfrac{dy}{dt} = -2$ (negative, because his distance from the house is *decreasing*). When $x = 120$ and $y = 50$, $s = 130$. So

$$130\frac{ds}{dt} = (120)(5) + (50)(-2), \quad \frac{ds}{dt} = \frac{50}{13} \approx 3.84615 \text{ feet per second.} \quad \square$$

---

20. A bird flies at a constant speed of 16 feet per second at a constant height of 48 feet. Its path takes it directly over a camera, which turns to track the bird. At what rate is the acute angle between the ground and the line of sight from the camera to the bird changing 4 second after it has passed above the camera?

Let θ be the acute angle between the ground and the line of sight from the camera to the bird, and let x be the distance the bird has flown past the point directly above the camera.

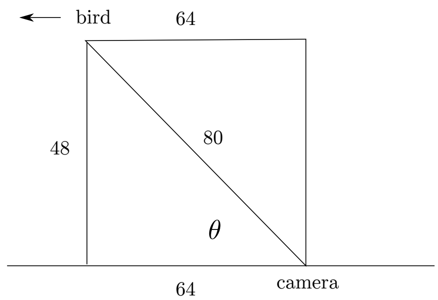


Then

$$\tan\theta = \frac{48}{x}$$

$$(\sec\theta)^2\frac{d\theta}{dt} = -\frac{48}{x^2}\frac{dx}{dt}$$

At 16 feet per second, the bird will have flown $x = 4 \cdot 16 = 64$ feet in 4 seconds. By Pythagoras, the hypotenuse of the triangle is $\sqrt{48^2 + 64^2} = 80$. Hence, $\sec\theta = \dfrac{80}{64}$.

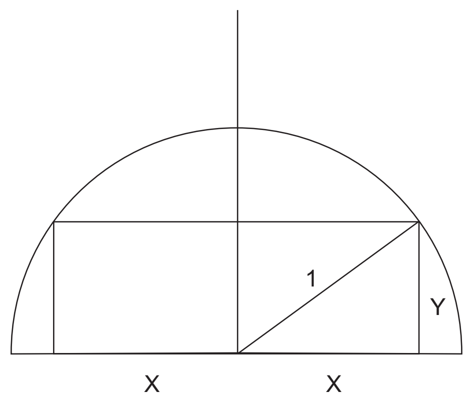


Thus,

$$\left(\frac{80}{64}\right)^2 \frac{d\theta}{dt} = \left(-\frac{48}{64^2}\right)(16)$$
$$\frac{d\theta}{dy} = -\frac{3}{25} \quad \square$$

---

21. Find the dimensions of the rectangle with the largest possible perimeter that can be inscribed in a semicircle of radius 1.



The height of the rectangle is y; the width is $2x$.
The perimeter of the rectangle is

$$p = 4x + 2y.$$

By Pythagoras, $x^2 + y^2 = 1$, so $y = \sqrt{1 - x^2}$. Therefore,

$$p = 4x + 2\sqrt{1 - x^2}.$$

$x = 1$ gives a "flat" rectangle lying along the diameter of the semicircle. $x = 0$ gives a "thin" rectangle lying along the vertical radius.

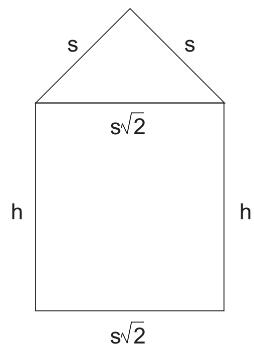
$\dfrac{dp}{dx} = 4 - \dfrac{2x}{\sqrt{1 - x^2}}$, so $\dfrac{dp}{dx} = 0$ for $x = \dfrac{2}{\sqrt{5}}$. (The negative root does not lie in the interval $0 \le x \le 1$.)

x	0	$\frac{2}{\sqrt{5}}$	1
p	2	$2\sqrt{5}$	4

$x = \dfrac{2}{\sqrt{5}}$ gives $y = \dfrac{1}{\sqrt{5}}$. The dimensions of the rectangle with the largest perimeter are $2x = \dfrac{4}{\sqrt{5}}$ and $y = \dfrac{1}{\sqrt{5}}$; the maximum perimeter is $p = 2\sqrt{5} \approx 4.47214$. □

---

22. A window is made in the shape of a rectangle with an isosceles right triangle on top.



(a) Write down an expression for the *total area* of the window.

$$A = \frac{1}{2}s^2 + \sqrt{2}sh. \quad \square$$

(b) Write down an expression for the *perimeter* of the window (that is, the length of the *outside* edge).

$$p = 2h + 2s + \sqrt{2}s. \quad \square$$

(c) If the perimeter is given to be 4, what value of s makes the total area a maximum?

Since $4 = p = 2h + 2s + \sqrt{2}s$,

$$h = 2 - s - \frac{\sqrt{2}}{2}s.$$

Hence,

$$A = \frac{1}{2}s^2 + \sqrt{2}s\left(2 - s - \frac{\sqrt{2}}{2}s\right) = \frac{1}{2}s^2 + 2\sqrt{2}s - \sqrt{2}s^2 - s^2 = 2\sqrt{2}s - \sqrt{2}s^2 - \frac{1}{2}s^2.$$

The extreme cases are $s = 0$ and $h = 0$, which gives $s = \dfrac{4}{2+\sqrt{2}}$.
The derivative is

$$\frac{dA}{ds} = 2\sqrt{2} - 2\sqrt{2}s - s.$$

Find the critical point:

$$0 = 2\sqrt{2} - 2\sqrt{2}s - s, \quad s = \frac{2\sqrt{2}}{2\sqrt{2}+1}.$$

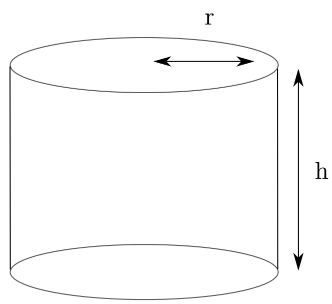
Plug the critical point and the endpoints into A:

s	0	$\dfrac{2\sqrt{2}}{2\sqrt{2}+1}$	$\dfrac{4}{2+\sqrt{2}}$
A	0	1.04482	0.68629

When $s = \dfrac{2\sqrt{2}}{2\sqrt{2}+1}$, the area is a maximum. $\square$

---

23. A cylindrical can with a top and a bottom is to be made with 96π square inches of sheet metal with no waste. What values for the radius r and the height h give the can of largest volume?



I have

$$V = \pi r^2 h \quad \text{and} \quad 96\pi = 2\pi r^2 + 2\pi r h.$$

Solving the second equation for h, I get

$$\begin{aligned} 96\pi &= 2\pi r^2 + 2\pi r h \\ 96\pi - 2\pi r^2 &= 2\pi r h \\ \frac{96\pi - 2\pi r^2}{2\pi r} &= h \end{aligned}$$

Plug this into V:

$$V = \pi r^2 \cdot \frac{96\pi - 2\pi r^2}{2\pi r} = \frac{1}{2} r(96\pi - 2\pi r^2) = 48\pi r - \pi r^3.$$

Since $r = 0$ is ruled out (as it causes division by 0 in the equation for h), I won't have two endpoints. I will use the Second Derivative Test. I have

$$\begin{aligned} \frac{dV}{dr} &= 48\pi - 3\pi r^2 \\ \frac{d^2V}{dr^2} &= -6\pi r \end{aligned}$$

Find the critical points by setting $\dfrac{dV}{dr} = 0$ and solving:

$$\begin{aligned} 48\pi - 3\pi r^2 &= 0 \\ 48 &= 3r^2 \\ 16 &= r^2 \\ 4 &= r \end{aligned}$$

(I can throw out $r = -4$, since the radius can't be negative.) This gives $h = \dfrac{96\pi - 32\pi}{8\pi} = 8$. In addition.

$$V''(4) = -24\pi < 0.$$

The critical point is a local max; since it's the only critical point, it's an absolute max. $\square$

---

24. (a) Find the absolute max and the absolute min of $y = x^3 - 12x + 5$ on the interval $0 \le x \le 5$.

The derivative is

$$y' = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

$y' = 0$ for $x = 2$ and $x = -2$; however, only $x = 2$ is in the interval $0 \le x \le 5$. y' is defined for all x.

x	0	2	5
y	5	-11	70

The absolute max is at $x = 5$; the absolute min is at $x = 2$. □

(b) Find the absolute max and the absolute min of $f(x) = \frac{3}{4}x^{4/3} - 15x^{1/3}$ on the interval $-1 \le x \le 8$.

$$f'(x) = x^{1/3} - 5x^{-2/3} = \frac{x - 5}{x^{2/3}}.$$

$f' = 0$ for $x = 5$ and f' is undefined for $x = 0$. Both points are in the interval.

x	-1	8	0	5
$f(x)$	15.75	-18	0	-19.23723

The absolute max is at $x = -1$ and the absolute min is at $x = 5$. □

---

25. Use a limit of a rectangle sum to find the exact area under $y = x^2 + 3x$ from $x = 0$ to $x = 1$.

Divide the interval $0 \le x \le 1$ up into n equal subintervals. Each subinterval has length $\Delta x = \frac{1}{n}$. I'll evaluate the function at the right-hand endpoints of the subintervals, which are

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}.$$

The function values are

$$\left(\frac{1}{n}\right)^2 + 3\left(\frac{1}{n}\right), \left(\frac{2}{n}\right)^2 + 3\left(\frac{2}{n}\right), \ldots \left(\frac{n}{n}\right)^2 + 3\left(\frac{n}{n}\right).$$

The sum of the rectangle area is

$$\sum_{k=1}^{n} \left[\left(\frac{k}{n}\right)^2 + 3\left(\frac{k}{n}\right)\right] \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{3}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n^2} \cdot \frac{n(n+1)}{2}.$$

The exact area is

$$\lim_{n\to\infty} \left(\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n^2} \cdot \frac{n(n+1)}{2}\right) = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}. \quad □$$

---

26. Given that $f(x) = x^5 + 4x^3 + 17$, what is $(f^{-1})'(22)$?

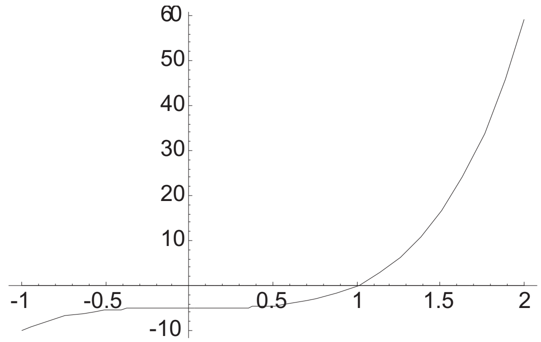
First, $f'(x) = 5x^4 + 12x^2$. Then

$$(f^{-1})'(22) = \frac{1}{f'(f^{-1}(22))}.$$

I need to find $f^{-1}(22)$. Suppose $f^{-1}(22) = x$. Then $f(x) = 22$, so

$$x^5 + 4x^3 + 17 = 22, \quad \text{and} \quad x^5 + 4x^3 - 5 = 0.$$

I can't solve this equation algebraically, but I can guess a solution by drawing the graph (of $y = x^5 + 4x^3 - 5$):



It looks like $x = 1$ is a solution. Check: $1^5 + 4 \cdot 1^3 - 5 = 0$.
Thus, $f(1) = 22$, so $f^{-1}(22) = 1$. Hence,

$$(f^{-1})'(22) = \frac{1}{f'(f^{-1}(22))} = \frac{1}{f'(1)} = \frac{1}{17}. \quad \square$$

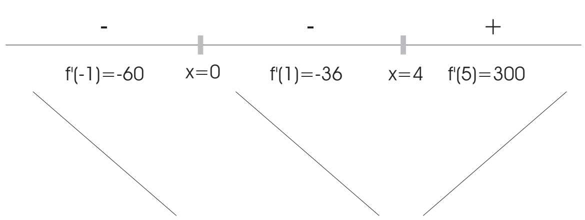
---

27. Find the largest interval containing $x = 1$ on which the function $f(x) = 3x^4 - 16x^3 + 1$ has an inverse f^{-1}.

The derivative is

$$f'(x) = 12x^3 - 48x^2 = 12x^2(x - 4).$$

$f'(x) = 0$ for $x = 0$ and $x = 4$. $f'(x)$ is defined for all x. Here is the sign chart for y':



f decreases for $x \le 4$, and this is the largest interval containing $x = 1$ on which f is always increasing or always decreasing. Therefore, the largest interval containing $x = 1$ on which the function $f(x) = 3x^4 - 16x^3 + 1$ has an inverse is $x \le 4$. $\square$

---

28. (a) Compute $\displaystyle \frac{d}{dx} \int_4^{x^6} \sqrt{2 + t^2}\, dt$.

$$\frac{d}{dx} \int_4^{x^6} \sqrt{2 + t^2}\, dt = \frac{dx^6}{dx} \frac{d}{dx^6} \int_4^{x^6} \sqrt{2 + t^2}\, dt = (6x^5)\sqrt{2 + (x^6)^2} = 6x^5 \sqrt{2 + x^{12}}. \quad \square$$

(b) Compute $\displaystyle \int \left(\frac{d}{dx} \sqrt[3]{x^2 + 1} \right) dx$.

$$\int \left(\frac{d}{dx} \sqrt[3]{x^2 + 1} \right) dx = \sqrt[3]{x^2 + 1} + C. \quad \square$$

29. (a) Use the definition of the derivative as a limit to prove that $\dfrac{d}{dx}\dfrac{1}{x-4} = -\dfrac{1}{(x-4)^2}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{x+h-4}-\dfrac{1}{x-4}}{h} = \lim_{h\to 0}\frac{\dfrac{x-4}{(x-4)(x+h-4)}-\dfrac{x+h-4}{(x-4)(x+h-4)}}{h} =$$

$$\lim_{h\to 0}\frac{\dfrac{(x-4)-(x+h-4)}{(x-4)(x+h-4)}}{h} = \lim_{h\to 0}\frac{\dfrac{-h}{(x-4)(x+h-4)}}{h} = \lim_{h\to 0}\frac{-h}{h(x-4)(x+h-4)} =$$

$$\lim_{h\to 0}\frac{-1}{(x-4)(x+h-4)} = -\frac{1}{(x-4)^2}. \quad \square$$

*(b) Compute $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\sqrt{t^4+1}\,dt$.

The limit as h goes to 0, the $\dfrac{1}{h}$, and the limits x and $x+h$ remind me of the definition of the derivative. So I make a guess that the limit in the problem is actually the derivative of a function. The problem is to figure out what function is being differentiated ....

Let $f(x) = \displaystyle\int_0^x \sqrt{t^4+1}\,dt$. Then

$$f(x+h)-f(x) = \int_0^{x+h}\sqrt{t^4+1}\,dt - \int_0^x\sqrt{t^4+1}\,dt = \int_0^{x+h}\sqrt{t^4+1}\,dt + \int_x^0\sqrt{t^4+1}\,dt = \int_x^{x+h}\sqrt{t^4+1}\,dt.$$

Therefore,

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\sqrt{t^4+1}\,dt = \lim_{h\to 0}\frac{1}{h}(f(x+h)-f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) =$$

$$\frac{d}{dx}\int_0^x\sqrt{t^4+1}\,dt = \sqrt{x^4+1}. \quad \square$$

30. Let

$$f(x) = \begin{cases} \dfrac{x+3}{6-x} & \text{if } x < 1 \\ 0.9 & \text{if } x = 1 \\ 3x^2 - 2.2 & \text{if } x > 1 \end{cases}.$$

Is f continuous at $x = 1$? Why or why not?

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{x+3}{6-x} = \frac{4}{5} = 0.8.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3x^2-2.2) = 3 - 2.2 = 0.8.$$

The left and right-hand limits agree. Therefore,

$$\lim_{x\to 1} f(x) = 0.8.$$

However, $f(1) = 0.9$, so $\displaystyle\lim_{x\to 1} f(x) \ne f(1)$. Hence, f is not continuous at $x = 1$. $\square$

31. (a) Suppose that $f(3) = 5$ and $f'(x) = \dfrac{x^2}{x^2 + 16}$. Use differentials to approximate $f(2.99)$ to 5 places.

Use

$$f(x + dx) \approx f(x) + f'(x)\, dx.$$

$dx = 2.99 - 3 = -0.01$ and $f'(3) = \dfrac{9}{25} = 0.36$. Therefore,

$$f(2.99) \approx f(3) + f'(3)\, dx = 5 + (0.36)(-0.01) = 4.99640. \quad \square$$

(b) A differentiable function satisfies $\dfrac{dy}{dx} = e^x \cos 3x$ and $y(0) = 0.1$. Use differentials to approximate $y(0.02)$.

$dx = 0.02 - 0 = 0.02$; when $x = 0$, $\dfrac{dy}{dx} = e^0 \cos 0 = 1$. Therefore,

$$dy = \frac{dy}{dx}\, dx = (1)(0.02) = 0.02.$$

Hence,

$$y(0.02) \approx y(0) + dy = 0.1 + 0.02 = 0.12. \quad \square$$

---

32. Use 3 iterations of Newton's method starting at $x = 2$ to approximate a solution to $4 - x^2 = e^x$.

Write the equation as $4 - x^2 - e^x = 0$. Set $f(x) = 4 - x^2 - e^x$, so $f'(x) = -2x - e^x$.

x	$f(x)$	$f'(x)$
2.	-7.38906	-11.3891
1.35121	-1.68789	-6.56454
1.09409	-0.183505	-5.17465
1.05863	-0.00311343	-4.99968
1.05801	-9.46553×10^{-7}	-4.99664

The solution is approximately $x \approx 1.058$. $\square$

---

33. Prove that the function $f(x) = x^3 + 2x - \cos x + 5$ has exactly one root.

First,

$$f(0) = 4 > 0 \quad \text{and} \quad f\left(-\frac{\pi}{2}\right) = -\frac{27\pi}{8} - \pi + 5 < 0.$$

By the Intermediate Value Theorem, f has a root between $-\dfrac{\pi}{2}$ and 0.

Suppose f has more than one root. Then f has at least two roots, so let a and b be two roots of f. By Rolle's theorem, f has a critical point between a and b.

However,

$$f'(x) = 3x^2 + 2 + \sin x.$$

Since $\sin x \geq -1$, $2 + \sin x \geq 1$. But $3x^2 \geq 0$, so

$$f'(x) = 3x^2 + 2 + \sin x \geq 0 + 1 = 1 > 0.$$

Thus, f has no critical points. Therefore, it can't have more than one root.
It follows that f has exactly one root. $\square$

---

34. Compute the following limits:

(a) $\displaystyle\lim_{x\to 1} \frac{2 - 2e^{x-1}}{\sin(x-1)}$.

$$\lim_{x\to 1} \frac{2 - 2e^{x-1}}{\sin(x-1)} = \lim_{x\to 1} \frac{-2e^{x-1}}{\cos(x-1)} = \frac{-2}{1} = -2. \quad \square$$

(b) $\displaystyle\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x - 3}{2x^3 - 6x^2 + x - 3}$.

$$\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x - 3}{2x^3 - 6x^2 + x - 3} = \lim_{x\to 3} \frac{3x^2 - 4x - 2}{6x^2 - 12x + 1} = \frac{27 - 12 - 2}{54 - 36 + 1} = \frac{13}{19}. \quad \square$$

(c) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$.

$$\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\dfrac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{-\dfrac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2x^{1/2} = 0. \quad \square$$

(d) $\displaystyle\lim_{x\to 0} \frac{\cos 2x}{x^2 + 3x + 1}$.

$$\lim_{x\to 0} \frac{\cos 2x}{x^2 + 3x + 1} = 1.$$

L'Hôpital's rule doesn't apply, because plugging in $x = 0$ gives 1, which is not an indeterminate form. $\square$

(e) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$.

Set $y = \left(1 + \dfrac{3}{x}\right)^{2x}$, so

$$\ln y = \ln\left(1 + \frac{3}{x}\right)^{2x} = 2x\ln\left(1 + \frac{3}{x}\right).$$

Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 2x\ln\left(1 + \frac{3}{x}\right) = 2\lim_{x\to\infty} x\ln\left(1 + \frac{3}{x}\right) = 2\lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{3}{x}\right)}{\dfrac{1}{x}} =$$

$$2\lim_{x\to\infty} \frac{\left(\dfrac{1}{1 + \dfrac{3}{x}}\right)\left(-\dfrac{3}{x^2}\right)}{\left(-\dfrac{1}{x^2}\right)} = 6.$$

So

$$\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x} = e^6. \quad \square$$

(f) $\lim_{x\to+\infty}\left(\sqrt{x^2+4x}-x\right)$

$$\lim_{x\to+\infty}\left(\sqrt{x^2+4x}-x\right)=\lim_{x\to+\infty}\left(\sqrt{x^2+4x}-x\right)\cdot\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x}=\lim_{x\to+\infty}\frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x}=$$

$$\lim_{x\to+\infty}\frac{4x}{\sqrt{x^2+4x}+x}=\lim_{x\to+\infty}\frac{4x}{\sqrt{x^2+4x}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to+\infty}\frac{4}{\frac{1}{x}\sqrt{x^2+4x}+1}=$$

$$\lim_{x\to+\infty}\frac{4}{\sqrt{\frac{x^2+4x}{x^2}}+1}=\lim_{x\to+\infty}\frac{4}{\sqrt{1+\frac{4}{x}}+1}=\frac{4}{2}=2.\quad\square$$

(g) $\lim_{x\to0}\frac{\sin 4x+\tan 5x}{x\cos 3x+12x}$.

$$\lim_{x\to0}\frac{\sin 4x+\tan 5x}{x\cos 3x+12x}=\lim_{x\to0}\frac{4\cos 4x+5(\sec 5x)^2}{-3x\sin 3x+\cos 3x+12}=\frac{9}{13}.\quad\square$$

(h) $\lim_{x\to0}\frac{x-x\cos x}{x\sin x+2x}$.

$$\lim_{x\to0}\frac{x-x\cos x}{x\sin x+2x}=\lim_{x\to0}\frac{1+x\sin x-\cos x}{x\cos x+\sin x+2}=\frac{1+0-1}{0+0+2}=0.\quad\square$$

(i) $\lim_{x\to0^+}\left(e^{2x}+x\right)^{1/x}$.

Set $y=\left(e^{2x}+x\right)^{1/x}$. Then

$$\ln y=\ln\left(e^{2x}+x\right)^{1/x}=\frac{1}{x}\ln\left(e^{2x}+x\right)=\frac{\ln\left(e^{2x}+x\right)}{x}.$$

Therefore,

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln\left(e^{2x}+x\right)}{x}=\lim_{x\to0^+}\frac{\frac{2e^{2x}+1}{e^{2x}+x}}{1}=3.$$

Hence,

$$\lim_{x\to0^+}\left(e^{2x}+x\right)^{1/x}=e^3.\quad\square$$

(j) $\lim_{x\to\infty}\frac{x^2+\ln x}{5x^2+x+1}$.

This is an indeterminate form of type $\frac{\infty}{\infty}$, so I can apply L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{x^2+\ln x}{5x^2+x+1}=\lim_{x\to\infty}\frac{2x+\frac{1}{x}}{10x+1}=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}}{10}=\frac{2}{10}=\frac{1}{5}.\quad\square$$

(k) $\lim_{x\to\infty}\left(1+\frac{5}{x^2}\right)^{3x^2}$.

Let $y=\left(1+\frac{5}{x^2}\right)^{3x^2}$. Then

$$\ln y=\ln\left(1+\frac{5}{x^2}\right)^{3x^2}=3x^2\ln\left(1+\frac{5}{x^2}\right).$$

Hence,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 3x^2 \ln\left(1 + \frac{5}{x^2}\right) = 3 \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{5}{x^2}\right)}{\dfrac{1}{x^2}} = 3 \lim_{x\to\infty} \frac{\left(\dfrac{1}{1 + \dfrac{5}{x^2}}\right)\left(-\dfrac{10}{x^3}\right)}{-\dfrac{2}{x^3}} = 15.$$

Therefore,

$$\lim_{x\to\infty} \left(1 + \frac{5}{x^2}\right)^{3x^2} = e^{15}. \quad \square$$

(l) $\lim_{x\to\infty} \left(\sqrt{x^2 + 8x} - x\right)$.

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 8x} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2 + 8x} - x\right) \cdot \frac{\sqrt{x^2 + 8x} + x}{\sqrt{x^2 + 8x} + x} = \lim_{x\to\infty} \frac{x^2 + 8x - x^2}{\sqrt{x^2 + 8x} + x} =$$

$$\lim_{x\to\infty} \frac{8x}{\sqrt{x^2 + 8x} + x} = \lim_{x\to\infty} \frac{8x}{\sqrt{x^2 + 8x} + x} \cdot \frac{\dfrac{1}{x}}{\dfrac{1}{x}} = \lim_{x\to\infty} \frac{8}{\sqrt{1 + \dfrac{8}{x}} + 1} = \frac{8}{1 + 1} = 4. \quad \square$$

(m) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x}$.

Let $y = \left(1 + \frac{2}{x}\right)^{3x}$, so

$$\ln y = \ln\left(1 + \frac{2}{x}\right)^{3x} = 3x \ln\left(1 + \frac{2}{x}\right).$$

Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 3x \ln\left(1 + \frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{2}{x}\right)}{\dfrac{1}{3x}} = \lim_{x\to\infty} \frac{\left(\dfrac{1}{1 + \dfrac{2}{x}}\right)\left(-\dfrac{2}{x^2}\right)}{-\dfrac{1}{3x^2}} = 6 \lim_{x\to\infty} \frac{1}{1 + \dfrac{2}{x}} = 6.$$

Therefore,

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x} = \lim_{x\to\infty} y = e^{(\lim_{x\to\infty} \ln y)} = e^6. \quad \square$$

The best thing for being sad is to learn something. - MERLYN, in T. H. WHITE'S *The Once and Future King*